Teachers' Guide

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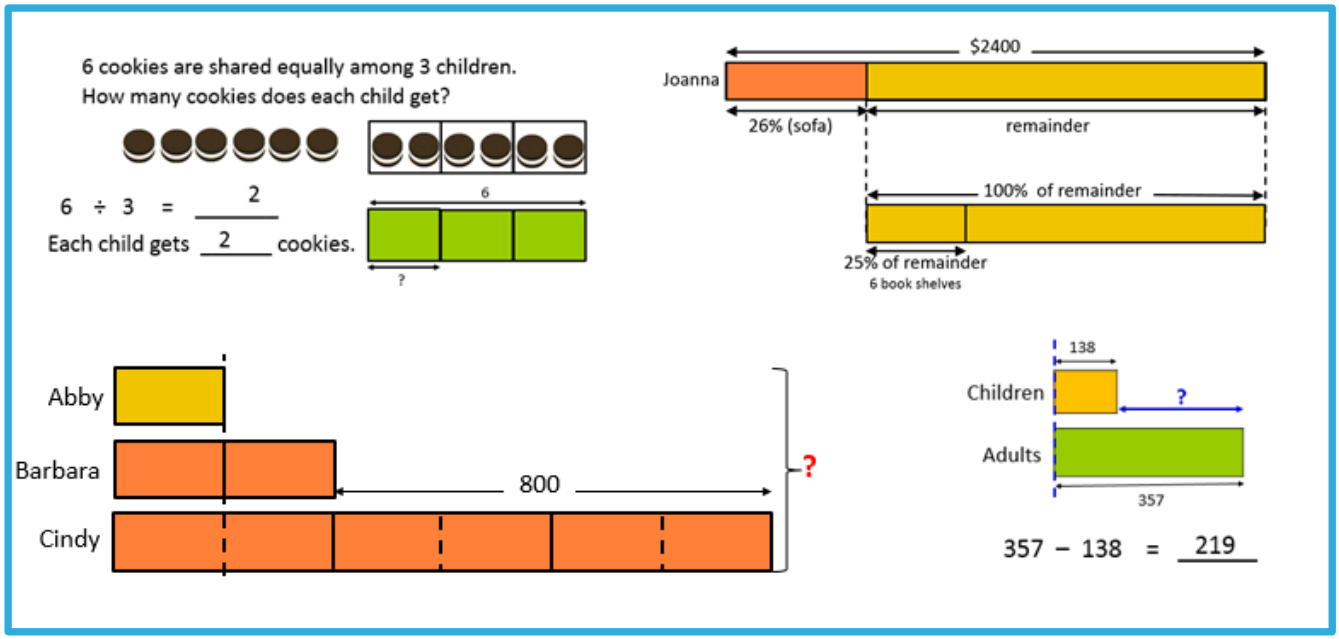

This special topic offers three units that introduce the bar model method that is widely used in Singapore, to complement Years 5 to 8 of the Australian Curriculum: Mathematics. The bar model method provides a visual strategy to help students unpack word problems, to visualise and organise information and to plan solutions. The intention is that students use the bar model method as a flexible problem-solving tool for all mathematical topics.

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About the Bar Model Method

The bar model method is a tool to assist problem solving. It is a key to the pedagogical approaches adopted in Singapore primary and lower secondary mathematics classrooms. The basic idea is that every quantity in a problem is represented by the length of a bar, and the bars are arranged to show the mathematical relationships between the quantities. Numerous worked examples are given in the units. Many good mathematical problem solvers create their own informal diagrams to show the data and the relationships between quantities, but the Ministry of Education in Singapore has supported considerable refinement of this method and promoted its use across most curriculum topics.

Teachers can use to learn the method by using these units, supported by the animated slideshows that demonstrate how to build the models step by step.

There are three main reasons for attention to the bar model method.

First, it provides a strong platform for students to display their mathematical thinking during problem solving, particularly in solving multi-step word problems. Students can represent the quantities and the relationships between them their relationships clearly using labelled bars, highlighting the knowns and unknowns. This supports students to devise a solution strategy, and to write abstract mathematical statements during the solution process.

Second, teachers can use the bar model in a dynamic manner to synthesise all the relevant problem information as the word problem is analysed, line by line, by students. This visual representation of the word problem reduces the cognitive load of students and especially helps lower attaining students.

Third, a bar model provides visual links between arithmetic thinking and algebraic thinking. It highlights the use of a variable in a pictorial way (i.e. an unknown quantity represented by the length of a bar or a common part of multiple bars) and shows relationships between the variable and other quantities. This helps students make sense of algebraic expressions and guides the solution of simple algebraic equations.

Four types of bar model

The three units include four types of bar models. The two basic types are the *part-whole* and *comparison* models and the *change* and *stack* models are common extensions of these. The diagrams below show simple examples. Each type of bar model has a clear purpose, making explicit different mathematical relationships between the quantities involved. Two or more types of bar models can be integrated to represent a more complex problem. There is flexibility within each type of bar model as to how the bars are arranged and labelled, to suit the problem situation.

Part-Whole Model

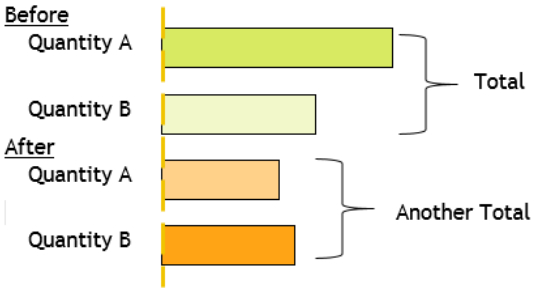
This model is for problems that involve one whole (i.e. a quantity) composed of or divided into two or more parts. We use the part-whole model to find one part given the whole and other parts, or the whole given the parts. Often, some of the parts are equal. The unknown quantity can be found using the arithmetic operations of addition, subtraction, multiplication and division.

Comparison Model

This model is for problems that involve comparing between two or more quantities, each of which is represented by a bar. The horizontal bars are typically vertically aligned to show the difference in quantities clearly. Students work out one or more of the unknown quantities (or differences between quantities) using the data that they have written on the model.

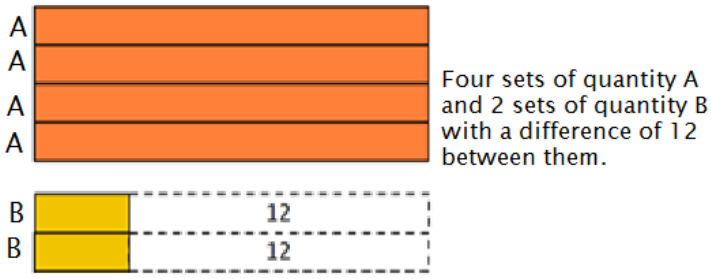
Change Model

This model is for problems that involve changes in one or more quantities during comparison, or changes in some parts within the whole. Most commonly, there are 2 sets of 2 bars.



Stack Model

This model is for problem situations that involve comparing between two or more quantities, with at least one of the quantities duplicated multiple times.



Goals and Pedagogy

The *reSolve: Maths by Inquiry* special topic on The Bar Model Method presents a unique suite of three units, designed to guide teachers and students. Each unit consists of 8 lessons with detailed teacher notes to accompany the selected word problems. Lessons are crafted to complement the mathematical and attitudinal goals for Australian teaching. These units are designed for the content and process goals of the Australian Mathematics Curriculum (Years 5 - 8).

The Bar Model Method has proven its effectiveness in helping students solve word problems since the 1980s in Singapore schools. Its power lies in the use of visualisation to help problem solvers analyse and represent the quantities given in the problem and their relationships, thereby fostering mathematical reasoning. Visual methods are especially powerful for students are not learning mathematics in their mother tongue. This is very common in Singapore.

The various types of bar models serve as anchors for discussion of the problem-solving process while weaving in relevant curriculum goals. In addition to careful sequencing, the three units also feature two spiral curriculum progressions: the C-P-A pedagogical approach and Polya's four steps for problem solving.

Curriculum Goals

- The bar model method is intended as a problem solving tool that students can use anywhere in mathematics.
- The method supports mathematical **understanding** of the word problem. It draws connections between the variables, quantities and their relationships as provided, and also represents these succinctly on the bars to help students visualise the connections better and plan further problem solving steps.
- The lessons provide opportunities for students to activate their content knowledge, concept understanding and computational skills pertaining to whole numbers, fractions, algebra, percentages and ratio in connection to the mathematical reasoning for problem solving. This supports development of conceptual understanding, especially of fraction ideas and multiplicative reasoning.
- The bar model method provides a platform to foster **fluency** in formal and informal computational skills. It reveals connections between a visual representation of the word problem and subsequent mathematical working steps.
- The method activates students' mathematical **reasoning** where they investigate efficient mathematical strategies, based on a deep understanding of the problem situation as they have represented with the model.
- The types of bar models taught in the three units are arranged to showcase the effectiveness of the use of bar models in mathematical understanding and problem solving.

Main Features of Pedagogy

The Concrete-Pictorial-Abstract (C-P-A) pedagogical approach.

The C-P-A approach is a common pedagogical approach adopted by Singapore primary and lower secondary school teachers in the teaching and learning of mathematics. The bar model method is an integral part of this. In the concrete phase of the C-P-A approach, students use concrete objects to make sense of the relationships between the quantities in the word problem. The teacher then scaffolds the student to move on to drawing rectangular bars as pictorial representations of these relationships, drawing connections between the concrete and pictorial phases of the C-P-A approach with respect to the situation in the word problem. Teachers will observe this transition in the first lessons of Unit 1. In Singapore this is used with students in the earliest year levels.

The pictorial phase transitions to the abstract phase as students progressively write more complete mathematical statements that represent the relationships shown. In the abstract phase, students come up with mathematical statements detailing the mathematical reasoning which springs from the bar model. Connections between the three phases in the C-P-A approach are strengthened by the bar model, and the model helps students plan their problem-solving steps.

Spiral Curriculum Progression

There are two main aspects to the spiral curriculum structure of the units – within the units and between them. The units begin with the two basic types of bar models (part-whole and comparison), and progress from foundation to advanced use, with later introduction of the change and stack models. The early lessons develop firm foundations in preparation for moving to more complex problems. Mathematical content knowledge is sequenced carefully within each unit so the complexity of use of the bar model is complemented by deeper content-based mathematical thinking as the unit progresses. Such sequencing allows teachers to have clear goals for each lesson.

From the first to the third units, the various bar models spiral in depth (i.e. complexity of model use tied to difficulty level of word problem) and scope (i.e. bar model applied to expanding inter-connected mathematical content). Each unit contains carefully selected examples for whole class discussion and several practice problems for targeted skill-building.

Due to the spiral curriculum progression, ideally students will meet the three units in order. However, provision is made at the start of the the later units for students who need a concise introduction. Teachers can also highlight the use bar models when they are teaching other topics to consolidate students' knowledge.

Aiming for Flexibility

The lessons proceed with class demonstration followed by individual or group practice. Teachers need to be flexible about the balance of demonstration and having students try the problems themselves. However, it is always important for teachers to discuss different ways of drawing the models for the same problem because the bar model becomes a more useful tool when students see different ideas that work. In addition, class discussion can highlight why some models are inappropriate and identify errors in overall strategy. The overall intention is to provide students with a flexible problem solving tool.

Polya's Four Steps for Problem Solving

Polya's four problem solving steps are embedded in Units 2 and 3 with the intention that they also become a flexible problem solving tool for students. The mathematician Georg Polya wrote a famous book "How to solve it?" (Polya, 1945) which outlined four very general steps to guide students' problem solving. Using these steps avoids students either jumping in to a solution without thought or giving up too quickly. The four steps are a useful guide but are not prescriptive because problem solvers often need to move back and forth between the steps as they adjust their approach. Suggested questions and prompts are provided within each lesson plan in Units 2 and 3 to scaffold students' use of the four steps, weaving it in with the use of appropriate bar models.

Polya's Steps	Focus
Understand	• Interpreting the context • Identifying the given quantities • Stating and clarifying the unknown quantity or quantities to be found • Establishing relationships (e.g. additive, multiplicative) between all quantities • Articulating the essential conditions and data (stated or implicit) in the problem statement
Plan	• Deciding on the type of bar model to be drawn (this may take several tries) • Outlining how the bar model can be drawn, labelling the model with relationships (e.g. part-whole, differences) between quantities explicitly shown • Marking out the unknown quantity or quantities to be found on the bar model • Writing abstract mathematical statements arising from the bar model which lead to the appropriate choice of operations (e.g. addition, subtraction,) to work out the answer
Do	• Executing the plan • Monitoring its success at each stage, and if necessary revising the plan
Check	• Checking whether the answer is reasonable • Checking if the answer exactly satisfies all the conditions given in the problem • Exploring other possible solutions

Unit Outlines

Unit 1: Introduction to Bar Model Method (Year 5)

This unit introduces students to the various types of models for solving mathematics word problems. The main objective is to get students to appreciate the use of bar models to visualize the mathematical quantities and relationships within a problem, thus reducing the cognitive load in handling such tasks. The transition from a concrete to a pictorial approach is strong in the first lessons.

The lessons in this unit are sequenced largely using the types of bar models as an anchor because bar models take the foreground as the key instructional focus. The lessons are best taught in sequence, although it can be good to spread them over two terms. Much of this unit can be readily adapted for Years 3 and 4, if desired.

Unit 1: Introduction to Bar Model Method (Year 5)		
Lesson Name	**Content**	**Bar model types** **Number types**
Lesson 1: Addition of Whole Numbers	Students are introduced to the bar model method. They learn how to construct the part-whole bar model to represent addition contexts through worked examples with simple numbers and tasks with larger numbers. Three types of addition contexts are used.	Part-whole models Whole Numbers
Lesson 2: Subtraction of Whole Numbers	Part-whole bar models are used to represent subtraction problems. Students study different subtraction situations such as 'take away' and 'difference between'.	Part-whole models Whole Numbers
Lesson 3: Multiplication & Division	Students use bar models to represent multiplication and division contexts. The tasks involve equal groups multiplication and both partition and quotition variations of division situations.	Part-whole models Whole Numbers
Lesson 4: Addition of Fractions	Students create bar models to represent word problems involving fractions. They encounter situations where the bars that make up the model simultaneously represent an absolute number (e.g. $50) and a fraction of a specified quantity (quarter of a cost).	Part-whole models Fractions
Lesson 5: Subtraction of Fractions	Students use part-whole bar models to represent subtraction contexts involving fractions. The bar models provide support for intuitive methods of solving these problems, and this supports the development of fraction concepts and skills.	Part-whole models Fractions
Lesson 6: Comparison Model for Addition	The comparison model is introduced for problems that involve quantities with known differences (e.g. a quantity is 10 less than another; a quantity is one fifth larger than another). The bar models support an intuitive approach to any fraction calculations.	Comparison models Whole Numbers Fractions
Lesson 7: Comparison Model for Subtraction	The comparison model is used for subtraction problems. Variation comes from changing which quantity (larger, smaller and difference) is to be found, and from expressing the comparison using different language (e.g. fewer than, increased to).	Comparison models Whole Numbers
Lesson 8: Comparison Model for Multiplication	Students learn how the comparison bar model can display the mathematical relationships in problems involving multiplication and division (including both partition and quotition situations).	Comparison models Whole Numbers

Unit 2: Bar Models in Problem Solving (Year 6)

This unit builds on students' familiarity with the use of bar models to develop the capacity to handle challenging mathematics word problems. The complexity arises by having stories with more events, more quantities, and more complex relationships between them. The objective is to promote critical and creative thinking in the use of the bar model method to solve non-routine tasks, so that the method becomes a flexible tool. Polya's four steps are used to structure problem solving.

It is suggested that these lessons are taught in sequence, again possibly spread across terms. Ideally, the bar model method becomes a tool that students can use in all areas of mathematics, as is demonstrated in Unit 3.

Unit 2: Bar Models in Problem Solving (Year 6)		
Lesson Name	**Content**	**Bar model types Number types****
Lesson 1: Part-whole model - Division	Students study multi-step problems which need a combination of whole number arithmetic operations for their solution. They make two different types of bar models for different division situations. They structure their problem solving by Polya's four phases.	Part-Whole Models Whole Numbers
Lesson 2: Part-whole model - Fractions	Students use part-whole models to represent different real-world contexts involving fractions. Solving the word problems is supported visually by the bar model and structured by Polya's phases of problem solving.	Part-Whole Models Fractions
Lesson 3: Part-whole model - Fractions of fractions	Students encounter problems where the part-whole model is used to represent both fractions and fractions of those fractions. The bar models support an intuitive approach, building understanding of fractions and fraction calculations.	Part-Whole Models Fractions
Lesson 4: Comparison model - Whole numbers	Students use the comparison bar model to help solve multi-step word problems that include information about additive and multiplicative relationships between quantities.	Comparison Model Whole Numbers
Lesson 5: Comparison model - Fractions	Students solve word problems that give information about fractions of different quantities that represent equal amounts. They draw comparison models and solve the problems by identifying a common unit in the different quantities.	Comparison Model Fractions
Lesson 6: Stack Model	This lesson introduces the stack model. Information provided in the word problems compares two groups of quantities, with either a multiplicative or additive relationship given between individual items.	Stack model Whole Numbers
Lesson 7: Change Model - Whole Numbers	This lesson introduces the change bar model, a variant on the comparison model. Students construct visual representations of complex stories, usually involving multiple quantities that change over time, and use the representations to find a strategy for solving the problems.	Change models Whole Numbers
Lesson 8: Change Model - Fractions	Students use the change model with word problems where the quantities in a situation change either by a whole number amount, or by a fraction of the whole. By working with the visual model, fraction calculations are replaced by intuitive steps.	Change models Fractions

** Word problems involving decimals work like those with whole numbers.

Unit 3: Bar Model Method and Secondary Mathematics (Years 7 and 8)

This unit demonstrates how the bar model method can be used in secondary school mathematics topic as a tool for problem solving. Importantly the unit also promotes informal (and later formal) algebraic thinking, by assisting in setting up and solving equations for solving word problems.

These lessons are grouped according to mathematical topic (number operations, algebra, percentage and ratio). It is suggested that teachers use Lessons 1 and 2 initially, and then use the pairs of lessons on algebra, percentage or ratio when these topics occur in the normal school schedule. Lesson X provides a quick introduction to bar models for students who have not done the earlier units.

Unit 3: Bar Model Method and Secondary Mathematics (Years 7 and 8)		
Lesson Name	**Content**	**Bar model types**
Lesson X: Preparation for Bar Models	Learning basic use of bar models (quick introduction for students without prior experience)	Part-Whole and Comparison
Lesson 1: Varieties of Bar Models (Whole Numbers)	This lesson reviews and extends students earlier knowledge of bar models. Students represent multi-step, multi-part word problems, using a variety of bar model types.	Combinations of types of bar models.
Lesson 2: Varieties of Bar Models (Fractions)	Students use the bar model method as a tool to help solve word problems involving fractions, and fractions of fractions.	Part-whole, comparison, change.
Lesson 3: Algebra with Bar Models 1	Students use bar models to assist in forming and solving simple linear algebraic equations from word problems. They label unknown quantities using algebra, set up equations, then solve them, with the bar model as a supporting visual representation.	Combinations of types of bar models
Lesson 4: Algebra with Bar Models 2	Students set up bar models to represent problem situations, label quantities with algebraic variables and expressions, set up equations (all linear), and then solve them to find unknowns. Understanding the standard algebraic techniques is supported by the visual representation.	Change models Linear equations
Lesson 5: Percentage of a Percentage	Students draw models to help solve word problems that involve finding a percentage of a percentage of a quantity (e.g. finding a discount of 10% on a price already discounted by 25%). The bar models highlight which 'whole' they are using at each of step.	Combinations of types of bar models
Lesson 6: Comparing Percentages	These problems involve percentages of different 'wholes'. The lesson emphasises that a percent is always relative to the nominated 'whole'. The bar models support finding a conversion factor between the percentages of the different 'wholes'.	Mainly comparison and change models
Lesson 7: Bar Models and Ratios 1	Bar models are used to visualise and then solve simple practical problems involving a single ratio. They then solve more difficult problems involving two ratios, where the 'units' from one ratio must be subdivided to make them compatible with the other.	Mainly comparison and change models
Lesson 8: Bar Models and Ratios 2	Students work with two and three ratios of quantities, mainly both before and after an amount has either been transferred between the quantities or otherwise changed. Alternative solutions to the problems using fractions or algebra can be explored, optionally connecting these topics with ratio.	Mainly comparison and change models

Further Information

References and further reading

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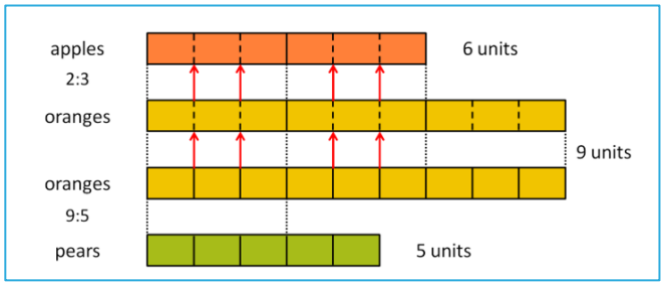
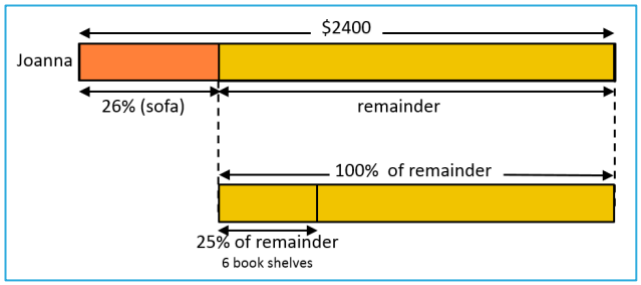
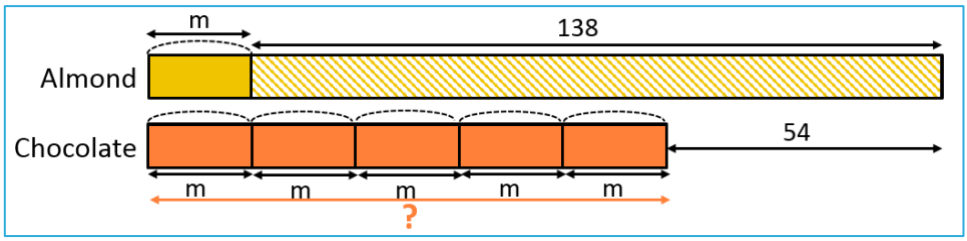


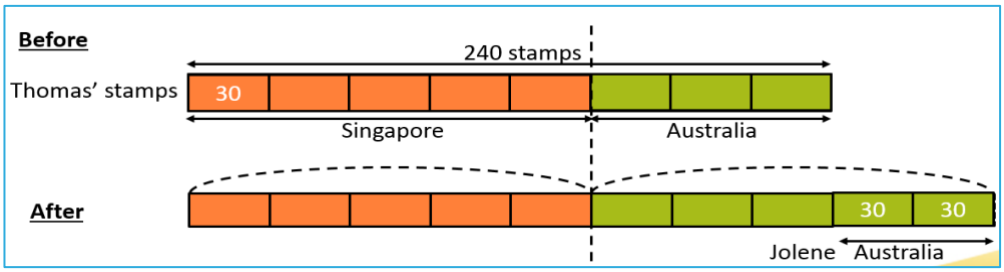
Bar model being used to solve a problem with two ratios.



Bar model for finding 25% of 74% of $2400.

Algebra bar model to find the number of almond cookies left after some sales.





Bar model to find a changed number of stamps.