INTRODUCTION

re(Solve) MATHS BY INQUIRY

Sum of Squares

Y7

Summary of learning goals

- The lesson is an extended investigation into a famous problem in mathematics. Students glimpse something of the history of mathematics, and how it can take centuries for mathematical questions to be finally decided. Students need to decide how they can record their work usefully (including their successful and unsuccessful approaches) and to work systematically to find patterns. They will see the importance of collecting evidence and organising it to show patterns, and discover the limitations of evidence for proving a mathematical result that holds for all numbers. The lesson also builds fluency in identifying perfect squares and, hence, in approximating square roots.

Australian Curriculum: Mathematics (Year 7)

ACMNA150: Investigate and use square roots of perfect square numbers.

- Investigating between which two whole numbers a square root lies.

Summary of lessons

Who is this sequence for?

- Students need very little mathematical content knowledge to undertake this lesson; however, they need to be able to square whole numbers and add and subtract them. The lesson calls upon, and will further develop, students' strategic skills for conducting an investigation, and the capacity to look for patterns and regularities and make conjectures. The lesson is structured to help students do these things.

Lesson 1: Diophantus and Lagrange

This inquiry explores the hypothesis of Diophantus, an ancient Greek mathematician, that any positive integer can be represented as the sum of four square numbers. Students explore the patterns that are generated by the sums of square numbers as they work systematically to rediscover and test the hypothesis. There are patterns of differing complexity to find, so the investigation is accessible to all. Finally, students use a number line to show between which two whole numbers a square root lies and consolidate their appreciation of the size of perfect squares.

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Reflection on this sequence

Rationale

This lesson is designed as a carefully structured investigation into pure mathematics. The structure supports students through the phases of mathematical problem-solving as they become acquainted with the mathematical ideas involved. They then immerse themselves into deep mathematical thinking and reflect on what has been found. The lesson is staged by first looking at examples and counter-examples, then assembling evidence systematically, all the while looking for patterns and conjectures and seeking reasons. Teachers may also choose to have students write a record of their work, to develop their communication capability.

Being able to conduct a substantial mathematical investigation is an important goal of the Australian Curriculum: Mathematics. This lesson gives students an experience of this with strong classroom support. Teachers can plan that, over time, students will undertake investigations with progressively less support and with more responsibility for deciding the best paths to take.

reSolve mathematics is purposeful

- By examining a mathematics problem of historical interest, students see mathematics as a living, breathing part of human society. They experience a substantial mathematical investigation, with many avenues to explore.
- Recognising perfect squares that can be used to sum to other numbers builds fluency and estimation skills.

reSolve tasks are inclusive and challenging

- The introductory activity is both accessible and intriguing. The start of the lesson develops a sense of curiosity whereby students wonder what the lesson is about. The obvious question that arises through the activity is why some numbers require three or four squares, yet others need only two. Explaining this leads to some interesting and challenging mathematics related to modulo arithmetic. Students can participate in mathematical activity at an appropriate depth for them, from conducting arithmetic trials systematically to pattern spotting and testing, whereas some will move towards deriving mathematical proofs that support the observations they make are always true.

reSolve classrooms have a knowledge-building culture

- The lesson is carefully designed to encourage students to display their results so that everyone in the class can see them. The use of sticky notes enables self-correction and provides the opportunity for students to improve upon the results of others.

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LESSON 1

re(Solve) MATHS BY INQUIRY

Sum of Squares

Diophantus and Lagrange

Y7

About this lesson

This inquiry explores the famous hypothesis of Diophantus, an ancient Greek mathematician, that any positive integer can be represented as the sum of four square numbers. Students explore the patterns that are generated by the sums of square numbers as they work systematically to rediscover and test the hypothesis. There are patterns of differing complexity, so the investigation is accessible to all.

Australian Curriculum: Mathematics (Year 7)

ACMNA150: Investigate and use square roots of perfect square numbers.

- Investigating between which two whole numbers a square root lies.

Mathematical purpose

- Through this extended investigation, students glimpse something of the history of mathematics, and how it can take centuries for mathematical questions to be finally decided. They need to decide how they can record their work usefully (including their successful and unsuccessful approaches) and to work systematically to find patterns.
- Students will see the importance of collecting evidence and organising it to show patterns, and discover the limitations of examples for proving a mathematical result that holds for all numbers. The lesson also builds fluency in identifying perfect squares and, hence, in approximating square roots.

Learning intention

- To investigate how positive whole numbers can be written as the sum of square numbers.

Time

Two lessons of approximately 1 hour each.

Vocabulary

- perfect squares
- square root
- squares

Resources

- large supply of sticky notes
- one single-sided copy of all pages from reSolve PDF *1c Sum of Squares Class Display* for display on wall
- one back-to-back copy of all pages from reSolve PDF *1b1 Sum of Squares Display Numbers* OR one single-sided copy of all pages of reSolve PDF *1b2 Sum of Squares Flip Cards*, cut vertically and folded into three.
- Student Sheet 1 – Class Recording Sheet (one per student)
- reSolve PowerPoint *1a Sum of Squares*
- reSolve PowerPoint *1d Sum of Squares Class Recording Sheet* for use on interactive whiteboard (optional)

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Before the lesson begins

Create a space for students to record their findings as a class.

Resources: Print the large recording sheets of reSolve PDF *1c Sum of Squares Class Display* onto A3 paper.

Display them on a wall, to make a table comprising eight columns and 15 rows (this will require 32 sheets of A3 paper and a large amount of space). Column 1 will then have the numbers 1, 9, 17, ..., 113, and column 8 will have the numbers 8, 16, 24, ..., 120.

Resources: Alternatively, plan to use the reSolve PowerPoint *1d Sum of Squares Class Recording Sheet* on an interactive whiteboard. Upon completion, this can be printed and handed out to students.

If neither of the above is possible, a large table can be drawn on an ordinary whiteboard.

Resources: If using flip cards (see reSolve PDF *1b2 Sum of Squares Flip Cards*) for the initial activity, they will need to be cut and folded in advance.

There are three identical flip cards on each A4 sheet. Cut them vertically so that each one has a number, its square and the diagram. Fold to make a tent shape without ends, so that each flip card can stand on the desk.

Initiate the inquiry: 46 as the sum of squares

Resources: Have ready the large printed numbers (reSolve PDF *1b1 Sum of Squares Display Numbers*) OR the already cut and folded flip cards (reSolve PDF *1b2 Sum of Squares Flip Cards*).

Write the number 46 in large writing on the whiteboard.

Ask a student (or, if doing the lesson in silence, beckon a student) to come to the front of the room and display the 36 card to the class.

Repeat with another student, giving them the 4 card.

Repeat six more times, giving out 1 cards to show:

36	4	1	1	1	1	1	1

Next to the number 46 on the board, write:

$$= 36 + 4 + 1 + 1 + 1 + 1 + 1 + 1$$

Now ask (or mime, if done silently) each student to turn over their cards so that the square numbers are displayed.

6^2	2^2	1^2	1^2	1^2	1^2	1^2	1^2

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Teacher notes:

- Writing the number directly on one side and as a square on the reverse provides a scaffold to make it easier for students to quickly identify perfect squares.
- The dot representation of square numbers shown on the flip cards reinforces the meaning of a perfect square.
- Cards involve students physically in the problem, enable results to be seen by the whole class, and encourage discussion and collaboration.

Write the following on the board, next to or underneath the previous equation:

$$= 6^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$$

Give a 2^2 card to another student. Ask the student to swap so that the squares still add to 46. (If conducting the activity in silence, show puzzlement, such as a chin rub, and use signs to suggest that the student could replace some of the 1 cards in line.)

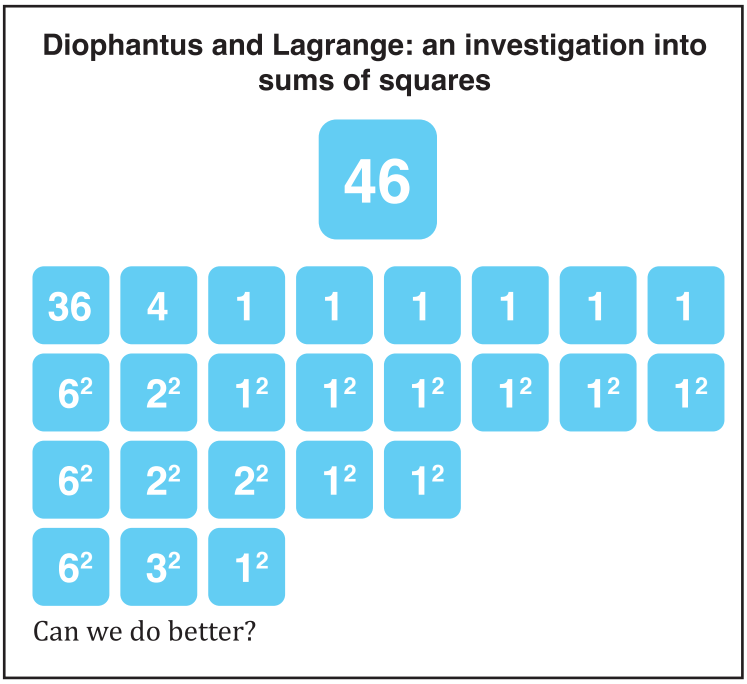
Write the following on the board, underneath the previous equation:

$$= 6^2 + 2^2 + 2^2 + 1^2 + 1^2$$

Repeat using a 3^2 card (to replace two 2^2 cards and one 1^2 card).

Write the following on the board, underneath the previous equation:

$$= 6^2 + 3^2 + 1^2$$



Ask: *Can we write 46 as the sum of squares using fewer than three squares? How about other numbers?*

Once students are satisfied that three is the smallest number of squares possible to write 46, place sticky notes showing 6^2, 3^2 and 1^2 in the 46 cell of the class recording sheet (or interactive whiteboard sheet or on the A4 sheet showing 46; see reSolve PowerPoint *1d Sum of Squares Class Recording Sheet*).

Teacher note:

- 46 was chosen deliberately, with an inefficient initial method, to promote discussion and to develop strategies that can be used to reduce the number of squares needed.

Resources: The procedure above is also shown on the reSolve PowerPoint *1a Sum of Squares*.

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Other numbers as sums of squares

Some more examples

Hand out all the number cards (reSolve PDF *1b1 Sum of Squares Display Numbers*) or the flip cards (reSolve PDF *1b2 Sum of Squares Flip Cards*), giving one to each student. (If you're short of cards, students can make their own similar card.)

Have the students display their card on their desk with the squared number showing (i.e. 5^2 showing rather than 25).

Now ask each student to write down a two-digit natural number. Choose a few of these as target numbers and ask the class to work together to reach the target using sums of squares. (Again, this could be done in silence, with inefficient methods being changed to more efficient ones.) As each number is completed, place sticky notes showing the square numbers used in the appropriate space on the class recording sheet or interactive whiteboard.

Once students are familiar with the problem and have tried a few examples, collect the cards. This is to encourage mental computation in the next phase of the lesson.

T **Teacher note:**

- A student will often start by using the largest square number that is less than the target number. Although this shows that they know the approximate size of its square root, this strategy will not always lead to the optimum result. There should be robust discussion of the smallest number of squares needed for each target number.

Pose the problem in its historical context

Show the picture of the cover of Diophantus' *Arithmetic, Book 6* from the reSolve PowerPoint *1a Sum of Squares* and also show where he lived (slides 8 and 9).

Diophantus of Alexandria was born around 200 CE and died around 290 CE. He is sometimes called the father of algebra. Diophantus investigated these questions and developed a hypothesis regarding the number of squares needed to write any natural number.

So what was Diophantus' hypothesis and how would you check to see if he was right?

Note that a hypothesis in mathematics is usually called a conjecture.

The intention here is that students should reconstruct Diophantus' hypothesis for themselves by gathering evidence about sums of squares.

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Strategies to write numbers as sums of squares

Do not, initially, hand out Student Sheet 1 – Class Recording Sheet. Rather, encourage students to work in groups and to add results to the class display. For example, if 10 groups are formed, each group could be asked to work on all the numbers ending with the digit of their group number.

Resources: At an appropriate time (which may vary between groups, depending on progress made) give each group a copy of Student Sheet 1 – Class Recording Sheet.

There are many ways for students to conduct their investigations: encourage them to choose a systematic approach, record the evidence well (including unsuccessful tries, such as when they cannot write a number as, say, a sum of three squares), and to record the observations that they use that make the searches easier. These ideas can be discussed.

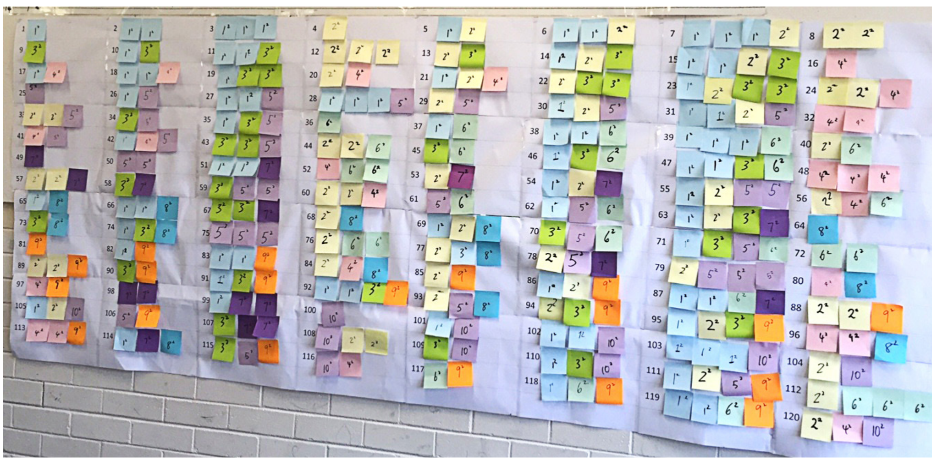
Teacher notes:

- It is not necessary for students to fill in each box in order commencing from 1. A better way may be to fill in perfect squares first, then realise that one more than a perfect square needs only two square numbers, as does nine more, etc.
- The eight-column structure helps students to identify patterns and make generalisations. For example, most of the integers requiring four squares are grouped in one column.

At a suitable time, pause the investigation and ask students to share any observations about which numbers can be made with only one square, two squares, etc. This may lead to a search to improve on results others have posted.

Some observations may include:

- Every natural number that is one more than a perfect square can be written using two square numbers.
- Numbers that are two more than a perfect square are sums of three squares.
- Numbers in columns 3 and 6 of the recording sheet (i.e. those of the form $8k + 3$; e.g. 3, 11, 19, … or $8k + 6$; e.g. 6, 14, 22, …) appear to require three squares.
- Numbers in column 7 of the recording sheet (i.e. those of the form $8k + 7$; e.g. 7, 19, 31, …) all require four squares.



A completed display board, showing 120 numbers in eight columns.

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Summary

Confirm that Diophantus' hypothesis is that every natural number can be written as the sum of not more than four squares. In the centuries since Diophantus, zero has also come to be recognised as a number. So, allowing zero, we can now state this as 'every natural number can be written as the sum of exactly four squares'. Lagrange proved in 1770 that Diophantus' hypothesis is true. Proving it, for absolutely every number, took mathematicians about 1500 years. Show the picture of the statue in Turin of Lagrange (slide 10 of reSolve PowerPoint *1a Sum of Squares*).

It is important to note that the class has collected a large quantity of empirical evidence regarding the number of squares needed to make natural numbers, but they have not found a proof.

In order to make the point that many examples do not prove that this hypothesis is true for all numbers, look at column 8. If I tried all the numbers only up to 100 (or even 111), I might think that three squares are enough. Only when I try 112 do I see that at least four squares are needed for that column. Who knows how many squares might be needed for even bigger numbers? We cannot tell from the examples but Lagrange's mathematical argument shows that four squares will always suffice.

It may be appropriate to discuss with students why numbers in columns 3 and 6 always need at least three squares, and numbers in column 7 need four. See the mathematical notes at the end of the lesson.

Going deeper

There are several ways to extend the inquiry. This is a good opportunity for students to choose an area to investigate further and they can later share their findings with the class. Suggestions include:

- Explore some numbers of choice beyond 120. Using technology will make this easier.
- Can some numbers be represented as the sum of squares in more than one way?
- Can some numbers that are themselves perfect squares also be written as the sum of two perfect squares? (*These are the squares of hypotenuses of right-angled triangles with integer sides; e.g. 25 = 16 + 9*). Or three perfect squares? (*These are the squares of diagonals of right-angled boxes with integer sides.*)
- How many perfect cubes do you need to add in order to make any natural number?
- Ask each student in the class to find out one piece of interesting information about Diophantus or Lagrange and share it with a group.

Fluency with square roots

This brief consolidation for fluency works to directly address the ACM elaboration; that is, 'investigating between which two whole numbers a square root lies'.

Students need to recognise that a number such as 46 lies between the two perfect squares 36 and 49; that is, 6 is the lower perfect square and 7 is the upper perfect square. Using mathematical symbols, this is presented as $6 < \sqrt{46} < 7$.

Check that students recognise the symbol for square root and its meaning.

Have a large number line marked 1 to 20 at the front of the room. Write $\sqrt{100}$ on a sticky note and ask students where it should be placed. Write $\sqrt{46}$ on a sticky note and ask students where it should be placed (i.e. between 6 and 7). Look at the fact that $\sqrt{100}$ is exactly 10 but $\sqrt{46}$ can be placed only approximately. Record $\sqrt{100} = 10$ and $6 < \sqrt{46} < 7$ on the board. Ask students to position other values, such as $\sqrt{176}$, $\sqrt{303}$ and $\sqrt{68}$, on the number line. Students should explain their reasoning for the placement of the sticky note. Their work with perfect squares in this lesson will have built familiarity with the concept of square numbers and with their sizes.

Encourage students to learn the squares of every natural number from 1 to 20. This is a useful extension of multiplication tables, and it also helps significantly when students learn Pythagoras' theorem.

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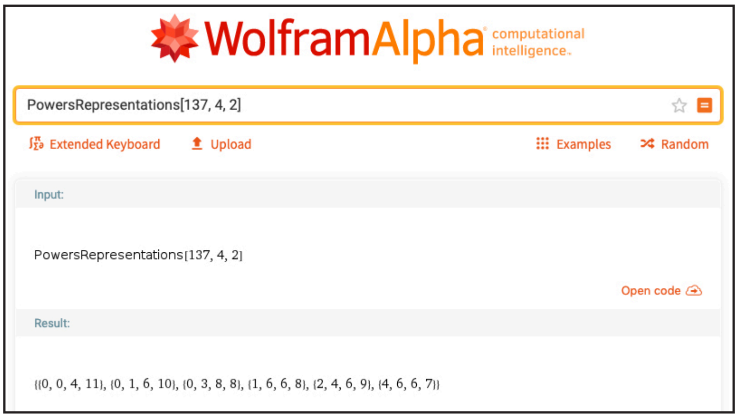
Technology

Wolfram Alpha (http://www.wolframalpha.com/) allows you to write any integer as a sum of any number of powers (when it is possible).

The expression to use is PowersRepresentations[n, k, p], where n is the integer, k is the number of terms in the sum and p is the power.

For example, typing in PowersRepresentations[137, 4, 2] gives the output screen below.



This means that there are six ways to write 137 as the sum of four squares:

$137 = 0^2 + 0^2 + 4^2 + 11^2$ $\quad 137 = 0^2 + 3^2 + 8^2 + 8^2$ $\quad 137 = 2^2 + 4^2 + 6^2 + 9^2$

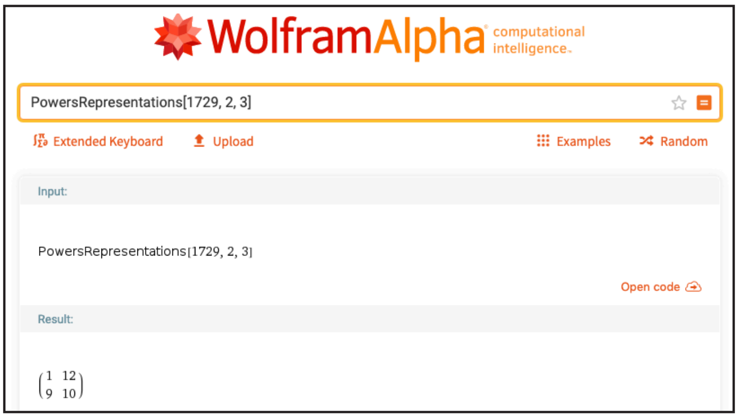
$137 = 0^2 + 1^2 + 6^2 + 10^2$ $\quad 137 = 1^2 + 6^2 + 6^2 + 8^2$ $\quad 137 = 4^2 + 6^2 + 6^2 + 7^2$

A very special number

A famous number is 1729, which is the number of the taxi that mathematician G.H. Hardy took to visit the Indian mathematician Ramanujan in hospital. The film *The Man Who Knew Infinity* (2014) is about Ramanujan's life. Hardy wrote:

> I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways'.

Entering PowersRepresentations[1729, 2, 3] into Wolfram Alpha gives the result:



This means that there are six ways to write 137 as the sum of four squares:

$1729 = 1^3 + 12^3$ $\quad 1729 = 9^3 + 10^3$

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Mathematical and historical notes

The arrangement in eight columns is deliberate because it displays useful patterns. The numbers in every column have the same remainder when divided by 8. So, for example, all numbers in the first column have remainder 1 when divided by 8. These numbers are said to be 'congruent to 1 modulo 8'. The squares of all numbers can also be calculated mod 8. For example, the squares of all numbers congruent to 3 mod 8 (e.g. 3, 11, 19, ...) are themselves all congruent to 1 mod 8.

Formulated mathematically, we can write:

In mod 8:

$0^2 \equiv 0^2$	$1^2 \equiv 1$	$2^2 \equiv 4$	$3^2 \equiv 1$
$4^2 \equiv 0^2$	$5^2 \equiv 1$	$6^2 \equiv 4$	$7^2 \equiv 1$

This means that perfect squares have a remainder of either 0, 1 or 4 when divided by 8.

A number in the first column (remainder of 1 when divided by 8) might be a perfect square itself.

A number in the second column (remainder of 2 when divided by 8) cannot be a perfect square itself. (Observe that 2 is not one of the squares in the list above.) To build it from perfect squares, there must be at least two squares congruent to 1 in the sum (and maybe some congruent to 0 or 4).

This means that all numbers of the form $8k + 2$ require at least two odd squares. Similarly, $8k + 3$ requires at least three odd squares; $8k + 6$ requires at least one even square (which will be congruent to 4 mod 8) and two odd squares (congruent to 1 mod 8); and $8k + 7$ requires at least one even square and three odd squares. This shows that all numbers of the form $8k + 7$ require at least four squares (but does not show that they can be made using only four squares).

Lagrange's 1770 proof that four squares is always enough involves two fundamental steps:

1. A proof that every prime number can be written as the sum of four squares.
2. A proof that the product of two numbers, each of which can be written as the sum of four squares, is itself the sum of four squares. This is called Euler's four-square identity.

The proof can be found in many places, including https://en.wikipedia.org/wiki/Lagrange%27s_four-square_theorem.

Lagrange was born in Italy, and his Italian name was Giuseppe Lodovico Lagrangia. He was one of the people who invented the metric system and introduced it during the time of the French Revolution.

In 1798, French mathematician Legendre proved that a natural number can be written as the sum of three squares if and only if it is not of the form $4^a(8b + 7)$ for integers a and b. Interestingly, this is harder to prove than Lagrange's four-square theorem, so it was proved later.

Also, in 1770, Edward Waring posed the problem of whether any natural number could be written as the sum of a fixed number of k^{th} powers of natural numbers. The fixed number would vary with k. For squares (i.e. $k = 2$), we know that the fixed number is 4.

Waring's conjecture was proved correct by Hilbert in 1909 and became known as the Hilbert–Waring theorem. The problem now has its own mathematics subject classification, used to classify papers in mathematics journals.

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Class Recording Sheet

Name: ______________________________

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104
105	106	107	108	109	110	111	112
113	114	115	116	117	118	119	120

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