AREA OF A CIRCLE

Lesson 1: About Circle Areas

Australian Curriculum: Mathematics (Year 8)

ACMMG197: Investigate the relationship between features of circles such as circumference, area, radius and diameter. Use formulas to solve problems involving circumference and area.

Lesson abstract

This lesson begins by approximating the area of a circle, between the area of an inscribed square and circumscribed square. Students next work in groups to understand one of four proofs of the exact formula $\mathrm{A} = \pi \mathrm{r}^2$ and explain their proof to others. In the process they refine both their own understanding and their explanations.

Mathematical purpose (for students)

The area of a circle can be proved to be exactly equal to πr^2.

Mathematical purpose (for teachers)

Students will first see a demonstration that the area of a circle is between $2r^2$ and $4r^2$. They will be given the opportunity to understand a student-accessible proof of the formula that $A = \pi r^2$. The range of proofs used gives teachers the opportunity to allocate students to a proof that is likely to be accessible to them.

Students explain their proofs to others because understanding is frequently deepened in this way. Students learn that mathematical results can be proved in ways which they can understand and that there can be multiple proofs. They engage with mathematical reasoning and develop skills in communicating it clearly. At the end of this lesson, students will be able to:

- Understand the circle area formula.

Lesson Length 60 minutes approximately

Vocabulary Encountered

- area
- formula
- proof
- sector
- polygon
- limit

Lesson Materials

- Student Sheet 1 – The Corner Square (2 pages, 1 for each student in the group)
- Student Sheet 2 – Slices of Pie (2 pages, 1 for each student in the group)
- Student Sheet 3 – Archimedes' Polygons (2 pages, 1 for each student in the group)
- Student Sheet 4 – Unravelling the Circle (2 pages, 1 for each student in the group)
- slide show 1a Area Intro powerpoint
- slide show 1b Area Images powerpoint

We value your feedback after this lesson via http://tiny.cc/lesson-feedback

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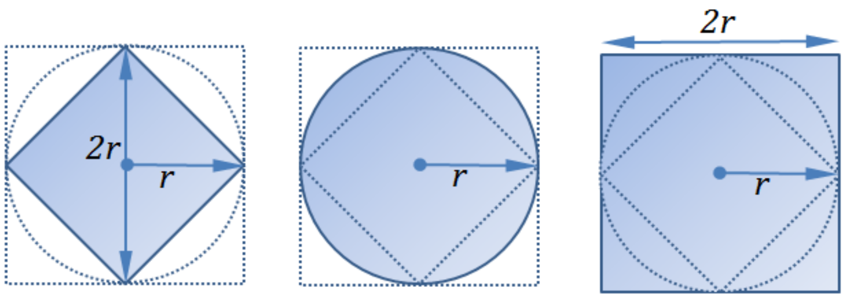
Getting Started

This introduction is designed to link with the approximate methods used for the circumference of a circle. It will provide a good introduction to the proofs.

Use the [1a Area Intro powerpoint]() slide show to demonstrate that the area of a circle of radius r lies between $2r^2$ (the area of a small square inscribed in the circle) and $4r^2$ (the area of a big square around the circle).

Small square area < Circle area < Big square area



$$2r^2 < \text{Circle area} < 4r^2$$

The slide show uses a circle of radius r, so that the big square has side length $2r$ and the small square has a diagonal of $2r$. For some classes, teachers may replace the variable r by a numerical value (e.g. $r = 10$).

The calculation of the area of the larger square is simple; the area of the smaller square can be found using the area of a triangle formula as in the slide show (or alternatively using Pythagoras' theorem if students know this).

When it has been shown that the area of the circle is between $2r^2$ and $4r^2$, students will probably be able to guess that the exact factor is π. This introduces the next section showing $A = \pi r^2$.

Looking at Proofs

The main task involves students working in small groups to understand a given proof and communicate this proof to others. The aim is that students, in their small groups, discuss their allocated circle area proof to understand it sufficiently well to explain it to the class.

Suggested Strategy

Arrange students into groups of 3 or 4 and allocate a proof to each group. Each student will need a copy of their groups' proof to facilitate discussion.

Phase 1 - Consensus building

- Small groups discuss their allocated proof to understand it fully, with teacher support as required.

Phase 2 - Small trial planning

- Plan a presentation within the small group to explain the argument to other students. To encourage students to use their own words, the presentations should use the 'blank' diagrams on the second page of the student sheets.

Phase 3 - Trial presentation

- Small group presents its argument to another small group, in order to convince them that $A = \pi r^2$.
- The audience group questions the presenters in order to fully understand the argument and to check its validity. They may make suggestions about the presentation.

Phase 4 - Presentation refinement

- Small groups refine their presentations, so that they can be given to the class.

Phase 5 - Whole class presentation (teacher selects one presentation per argument)

- Small group presents allocated proof to the whole class with an opportunity for others to ask questions so that everyone understands each proof.
- Discussion and comments are to focus on the reasoning, not the presentation.

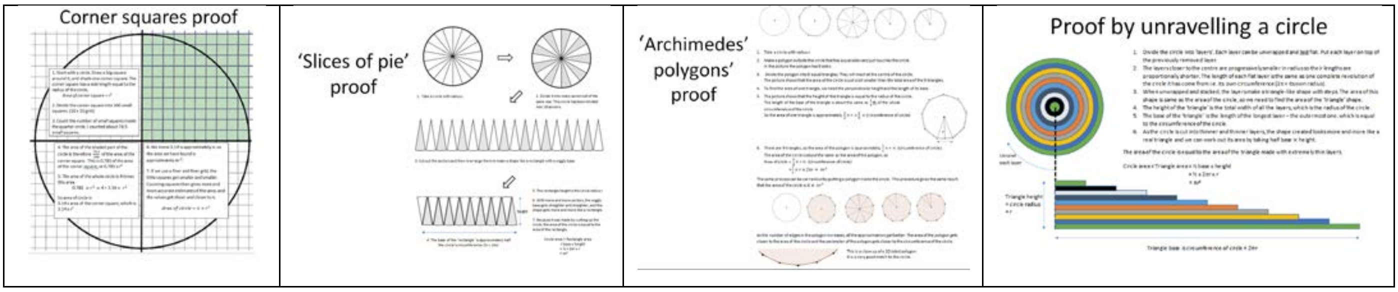
NOTE: slide show 1b Area Images powerpoint describes these phases and also contains the blank diagrams for student presentations.

Maintaining the inquiry

Students work through the 5 phases described above. The tasks of the teacher during this time are:

- Ensuring that students stay on task.
- Monitoring the progress of group discussion and clarifying for individual groups sections of the proofs that they do not yet understand.
- Hearing student attempts at explanation and asking relevant questions.
- To clarify that a limiting process is involved in all the proofs, but to handle it very gently.
- Time keeping. It is vital that all of the 5 phases happen.
- Bringing the class together for phase 5.
- Asking students to write in notebooks that $A = \pi r^2$, then glue in their proof sheet.
- If time permits ask, "Which idea was the most convincing for you? Why?"

Proof Summaries



Proof 1: The corner square

- Student Sheet 1 – The Corner Square
- A square grid is placed over and circumscribing a circle, and the squares inside one quadrant of the circle are counted (estimate the area of the partial squares).
- This number is multiplied by 4 to give the number of squares inside the whole circle.
- The number of squares inside one quarter of the square is found.
- Students connect the area of 'the corner square' (r^2) and the area of the whole circle.
- The relationship $A = \pi r^2$ is deduced.

This proof uses a numerical example, a ratio and then generalisation. For an easier approach, allocate the 'blank' version of this student sheet so that students can simply count the squares (314) and make the link to π.

Proof 2: Slices of pie

- Student Sheet 2 – Slices of Pie
- Students cut a circle into 18 equal sectors. They may need to actually do this (cut out and paste into place) or they may be content with the diagrams provided.
- Sectors are rearranged to approximate a rectangle (of side lengths r and πr).
- The area of the circle is equal to the area of the approximate rectangle so for both $A = \pi r^2$.

This proof uses visualisation (which may be aided by actually cutting and rearranging) as well as some generalisation.

Proof 3: Archimedes' polygons

- Student Sheet 3 – Archimedes' Polygons
- Circumscribed and inscribed 9-sided polygons (nonagons) are drawn for a circle of radius r.
- The circumscribed polygons are divided into 9 equal triangles with the circle centre as their common third point. For each triangle, the base is approximately $\frac{1}{9}$ of the circumference $2\pi r$ and the height is r.
- Using the formula for the area of a triangle the area of the circumscribed polygon is found to be approximately $9 \times \frac{1}{9}$ of $2\pi r \times r$ so for the polygon $A \approx \pi r^2$. Also for the circle $A \approx \pi r^2$.The area of the inscribed nonagon can be found in a similar way (but isn't).
- As the number of sides increases the polygons and the circle become more nearly the same, so with an infinite number of sides the areas will be identical and for the circle $A \approx \pi r^2$.

This proof uses visualisation, the idea of limits (handled very gently) and some generalisation.

Proof 4: Unravelling the circle

- Student Sheet 4 – Unravelling the Circle
- The circle is visualised as being composed of a series of concentric rings.
- It is cut along a radius and the rings become near rectangles that can be laid one above the other so that they form a right-angled triangle. Its base is the circumference of the circle, $2\pi r$, and its height is r.
- As the circle is divided into more and more rings, when they are cut, flattened and laid out they will form closer and closer to an exact triangle. Using the formula for the area of a triangle the area is calculated to be $A = \pi r^2$.

This proof uses visualisation, the idea of limits (handled very gently) and some generalisation.

Multiple proof reasoning

The aim is that students, in their small groups, discuss their allocated circle area proof to understand it sufficiently well to explain it to the class. Using multiple proofs will:

- Provide students opportunity to see a method that resonates with their understanding of circles and broader mathematics.
- Enable students to revisit the ideas underlying all the proofs multiple times.
- Provide opportunity for students to explain their proof to others who have not read it.

The arguments presented are called 'proofs' although they are not as rigorous as mathematicians require a proof to be. This is because each of these proofs requires a limiting process – the number of squares or sides or layers increases to infinity. The goal here is to present the main idea of the proofs in a way that students can understand.

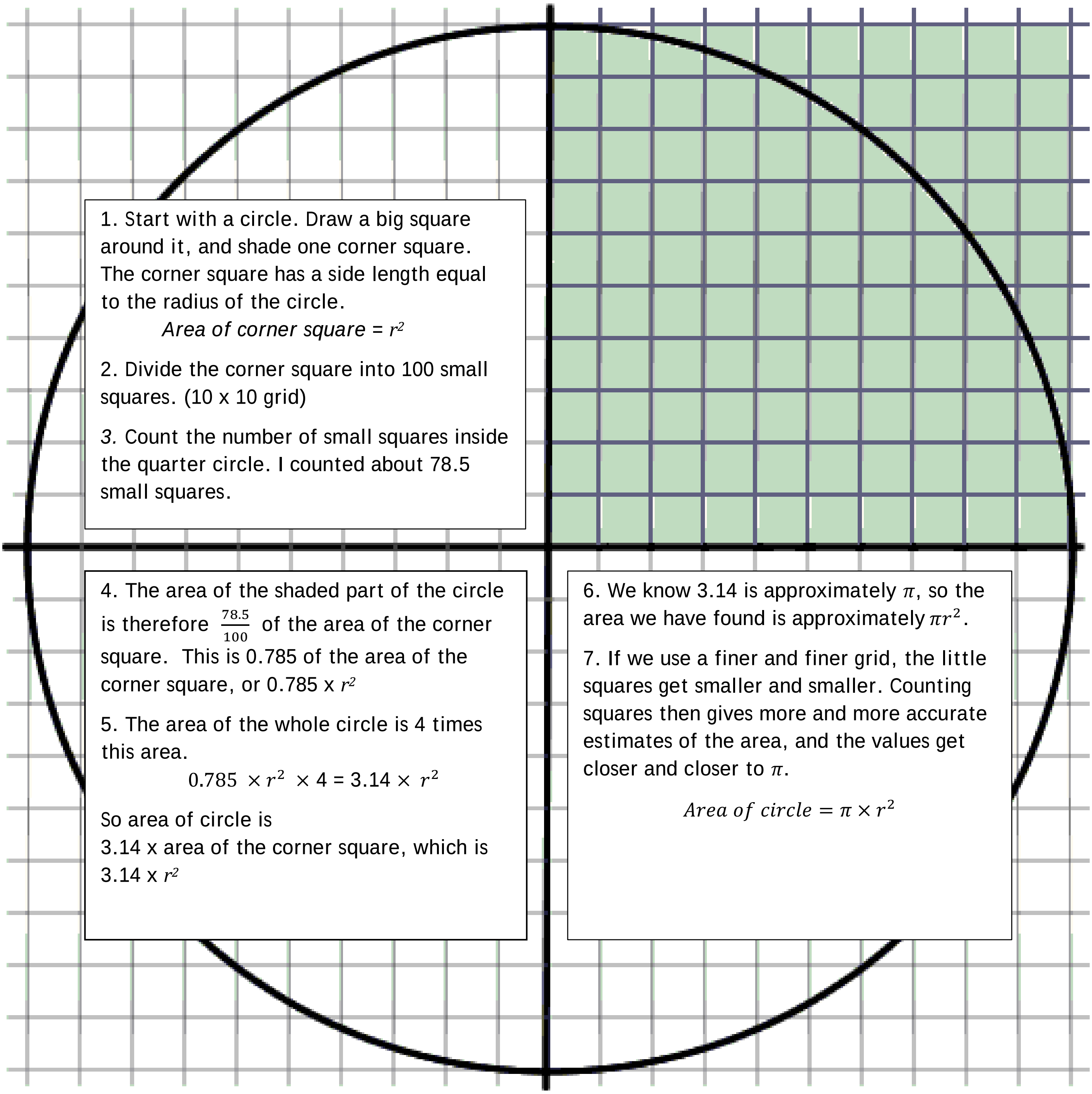
The phases allow students to gain individual and group understanding of their allocated proof, and confidence in explaining it. The teacher will need to monitor student understanding actively during the early phases.

To make the presentations less threatening to students, emphasise that the discussion and questions are to focus on the ideas and reasoning, rather than the presentation.

Students may find out that they did not properly understand their proof until the other group asked questions about it. This is an important role for the groups when listening. By explaining and having questions asked, the presenting students will understand their own proof more fully.

The Corner Square 1

Name: ______________



1. Start with a circle. Draw a big square around it, and shade one corner square. The corner square has a side length equal to the radius of the circle.

   Area of corner square = r^2

2. Divide the corner square into 100 small squares. (10 x 10 grid)

3. Count the number of small squares inside the quarter circle. I counted about 78.5 small squares.

4. The area of the shaded part of the circle is therefore $\frac{78.5}{100}$ of the area of the corner square. This is 0.785 of the area of the corner square, or 0.785 x r^2

5. The area of the whole circle is 4 times this area.

   $$0.785 \times r^2 \times 4 = 3.14 \times r^2$$

   So area of circle is
   3.14 x area of the corner square, which is
   3.14 x r^2

6. We know 3.14 is approximately π, so the area we have found is approximately πr^2.

7. If we use a finer and finer grid, the little squares get smaller and smaller. Counting squares then gives more and more accurate estimates of the area, and the values get closer and closer to π.

$$Area\ of\ circle = \pi \times r^2$$

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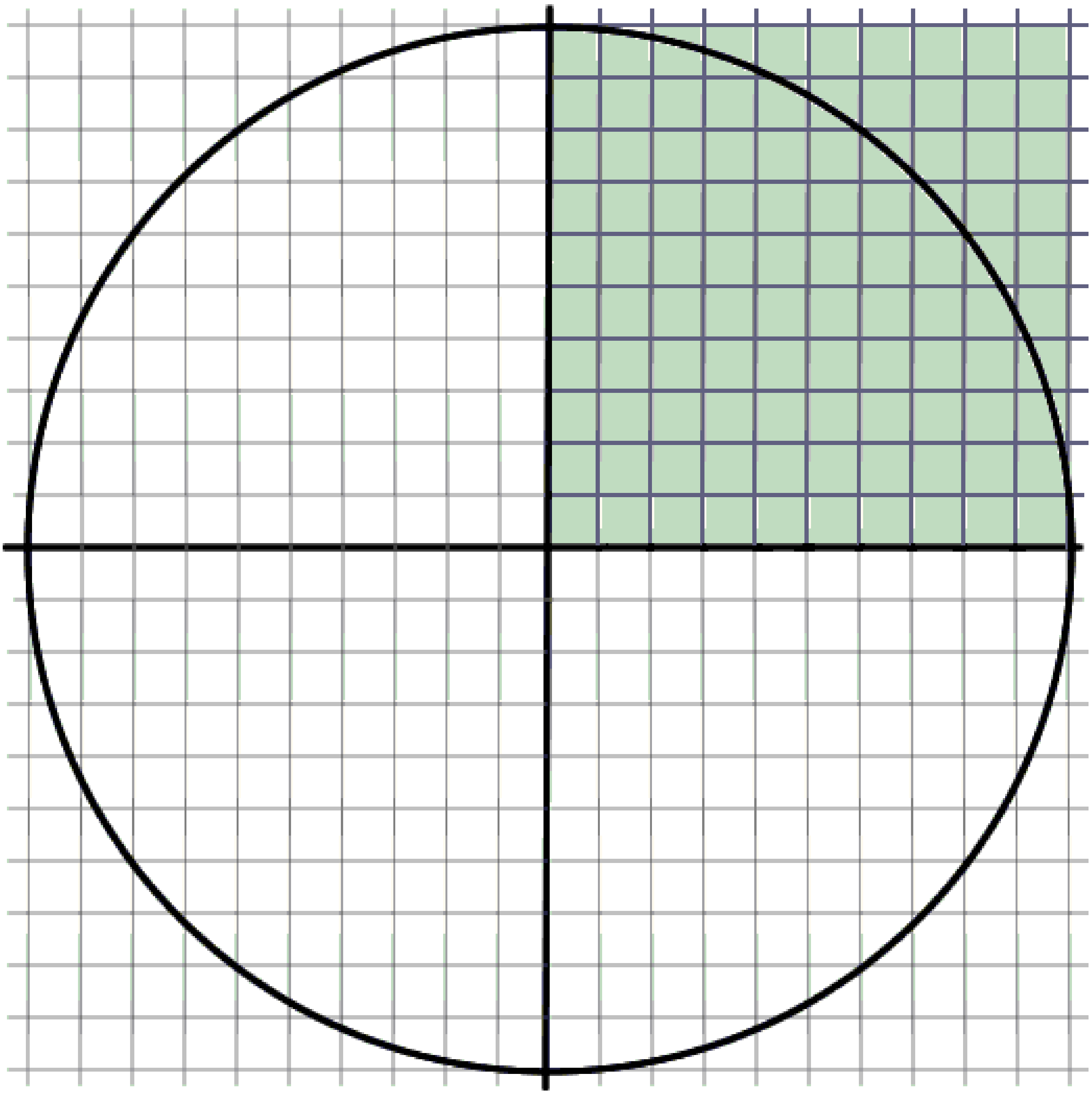
The Corner Square 2

Name: ______________



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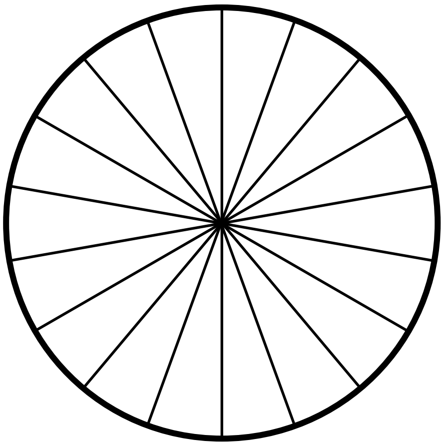
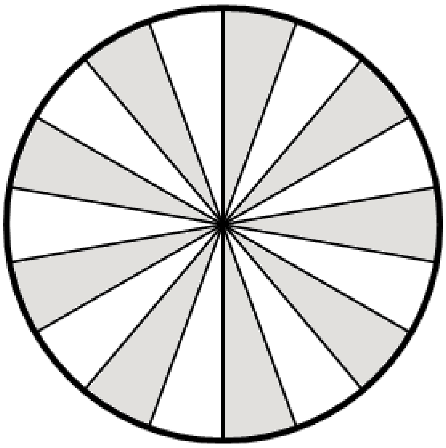
Slices of Pie 1

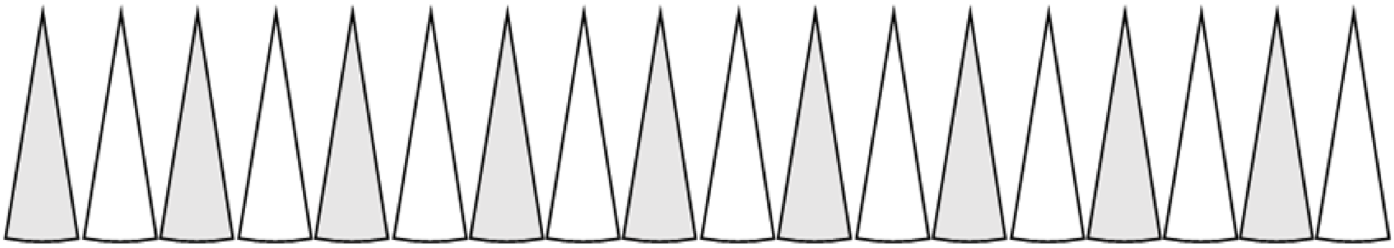
Name: ______________



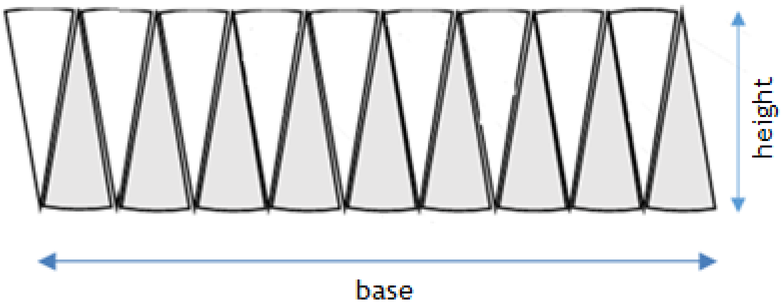


1. Take a circle with radius r.

2. Divide it into many sectors all of the same size. This circle has been divided into 18 sectors.



3. Cut out the sectors and then rearrange them to make a shape like a rectangle with a wiggly base.



4. The 'rectangle' height is the circle radius r.

5. The base of the 'rectangle' is approximately half the circle's circumference ($\frac{1}{2} \times 2\pi r$).

6. With more and more sectors, the wiggly base gets straighter and straighter, and the shape gets more and more like a rectangle.

7. Because it was made by cutting up the circle, the area of the circle is equal to the area of the rectangle.

Circle area = Rectangle area

= base x height

= $\frac{1}{2} \times 2\pi r \times r$

= πr^2

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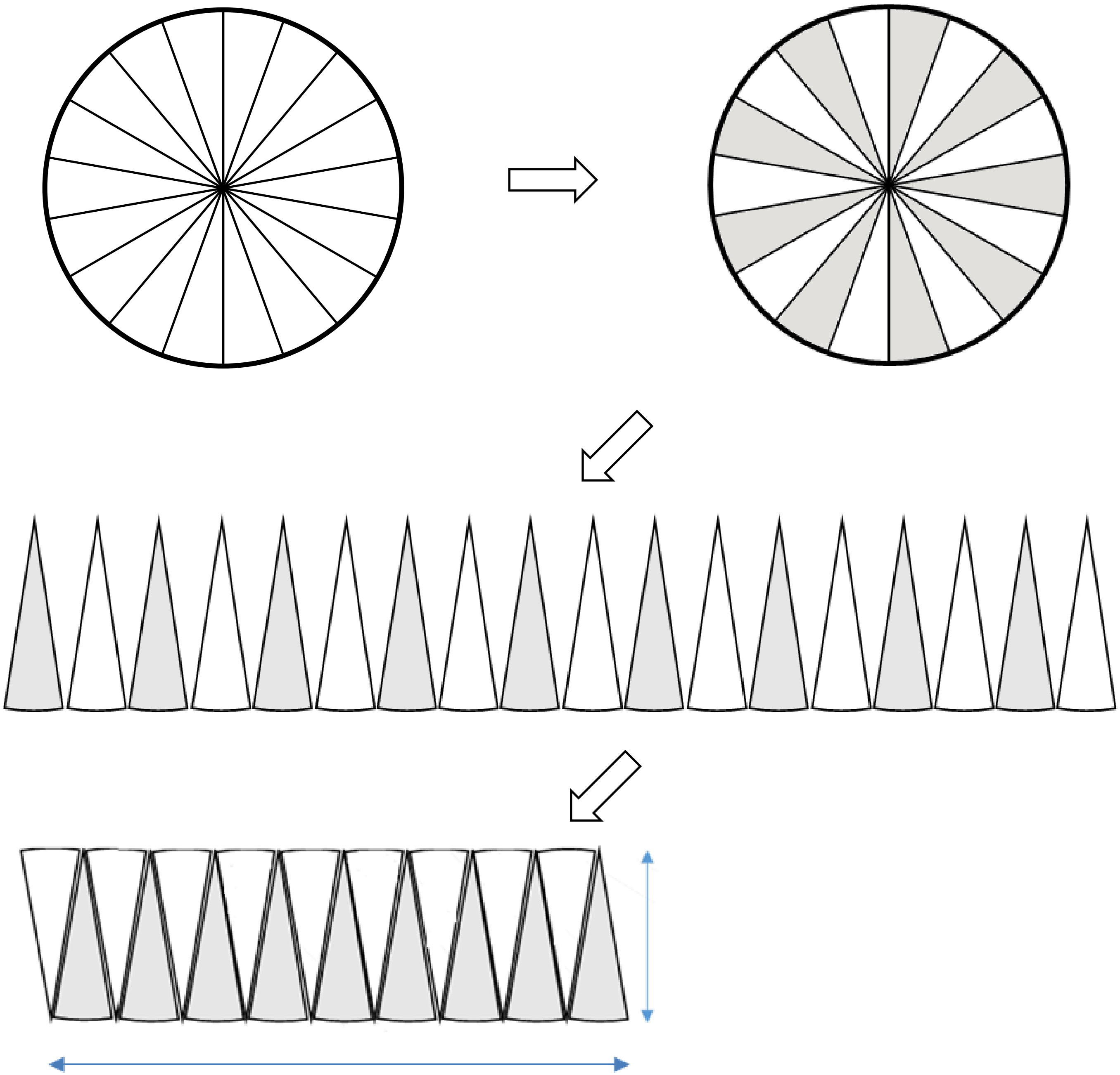
Slices of Pie 2

Name: ________________



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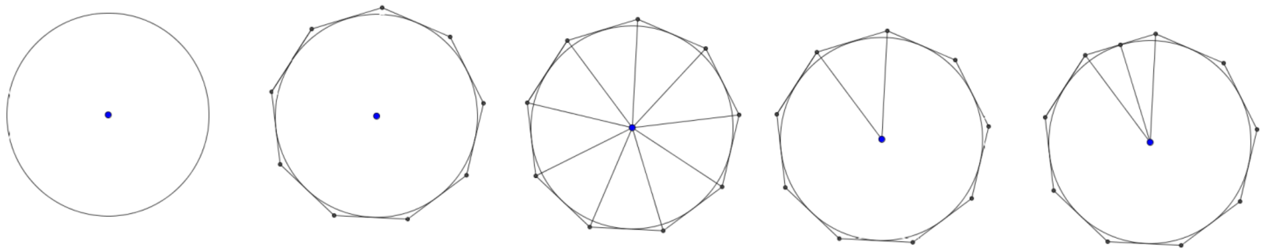
Archimedes’ Polygons 1

Name: ______________



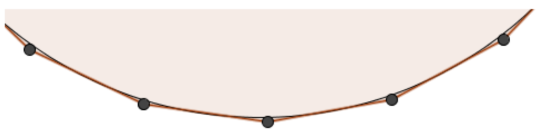
1. Take a circle with radius r.
2. Make a polygon outside the circle that has equal sides and just touches the circle.
   In the picture the polygon has 9 sides.
3. Divide the polygon into 9 equal triangles. They will meet at the centre of the circle.
   The picture shows that the area of the circle is just a bit smaller than the total area of the 9 triangles.
4. To find the area of one triangle, we need the perpendicular height and the length of its base.
5. The picture shows that the height of the triangle is equal to the radius of the circle.
   The length of the base of the triangle is about the same as $\frac{1}{9}$ th of the whole circumference of the circle.
   So the area of one triangle is approximately $\frac{1}{2} \times r \times \frac{1}{9} \times$ *(circumference of circle)*.
6. There are 9 triangles, so the area of the polygon is approximately $\frac{1}{2} \times r \times$ *(circumference of circle)*.
7. The area of the circle is about the same as the area of the polygon, so:

   Area of circle ≈ $\frac{1}{2} \times r \times$ *(circumference of circle)*

   $= \frac{1}{2} \times r \times 2\pi r = \pi r^2$

The same process can be carried out by putting a polygon inside the circle. This procedure gives the same result: that the area of the circle is $A = \pi r^2$

As the number of edges in the polygon increases, all the approximations get better. The area of the polygon gets closer to the area of the circle and the perimeter of the polygon gets closer to the circumference of the circle. The picture below is a close up of a 20-sided polygon showing that it is a very good match to the circle.



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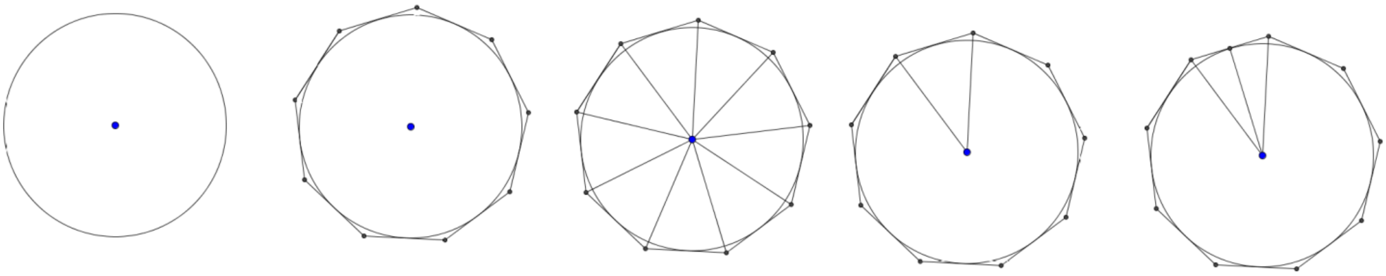
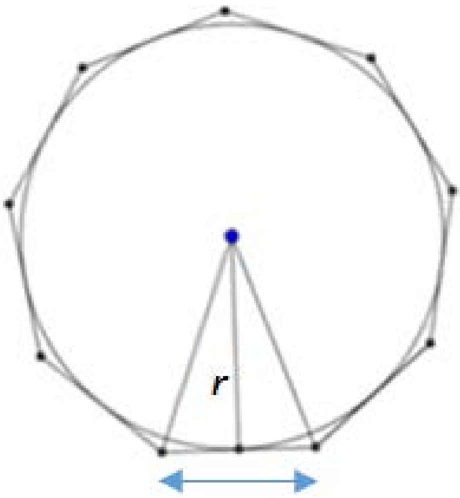
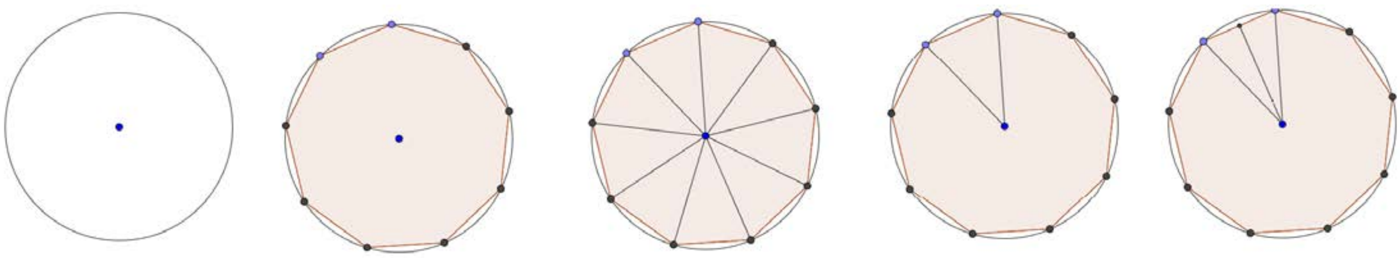
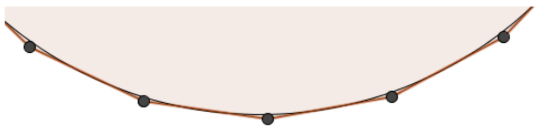
Archimedes' Polygons 2

Name: ____________









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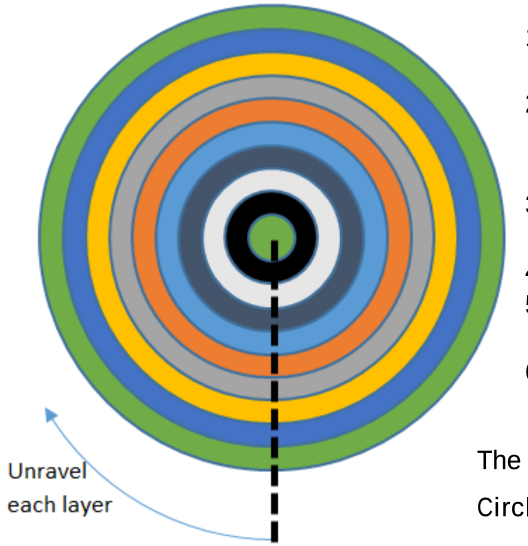
Unravelling the Circle 1

Name: ______________

1. Divide the circle into ‘layers’. Each layer can be unwrapped and laid flat. Put each layer on top of the previously removed layer.
2. The layers closer to the centre are progressively smaller in radius so their lengths are proportionally shorter. The length of each flat layer is the same as one complete revolution of the circle it has come from i.e. its own circumference (2π × its own radius).
3. When unwrapped and stacked, the layers make a triangle-like shape with steps. The area of this shape is same as the area of the circle, so we need to find the area of the ‘triangle’ shape.
4. The height of the ‘triangle’ is the total width of all the layers, which is the radius of the circle.
5. The base of the ‘triangle’ is the length of the longest layer – the outermost one, which is equal to the circumference of the circle.
6. As the circle is cut into thinner and thinner layers, the shape created looks more and more like a real triangle and we can work out its area by taking half base × height.

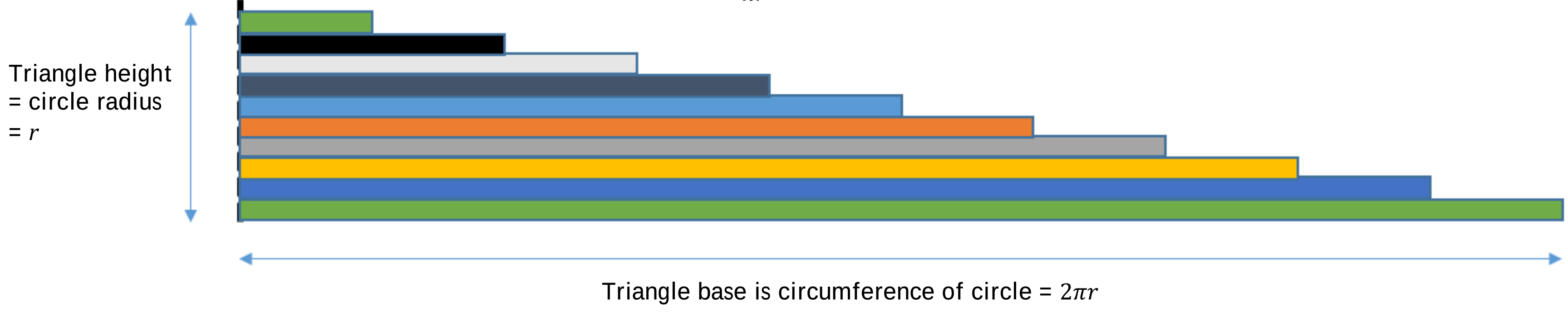


The area of the circle is equal to the area of the triangle made with extremely thin layers.

Circle area = Triangle area = ½ base x height

= ½ x $2\pi r$ x r

= πr^2



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Unravelling the Circle 2

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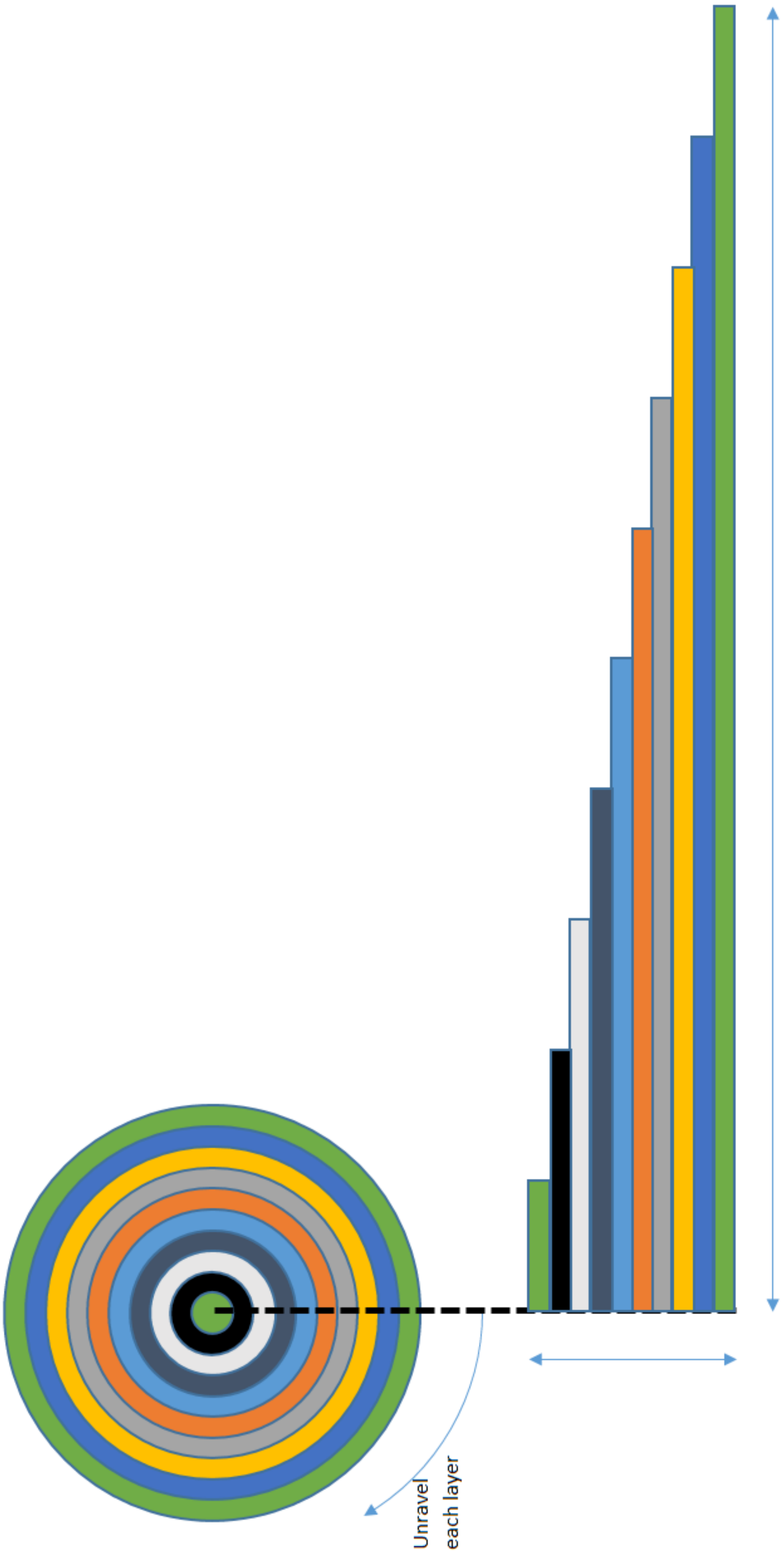

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