re(Solve) MATHS BY INQUIRY

Addition Chain

Y8

Summary of learning goals

- This resource focuses on mathematical reasoning using algebra. Students use spreadsheets to investigate potential arithmetic relationships and then use algebra to identify and justify which relationships are generally true. The task can be used as a springboard for an in-depth exploration of the Fibonacci sequence, and develops skills in using spreadsheets.

Australian Curriculum: Mathematics (Year 8)

ACMNA191: Factorise algebraic expressions by identifying numerical factors.

ACMNA192: Simplify algebraic expressions involving the four operations.

Summary of lessons

Who is this sequence for?

- This lesson is designed to consolidate skills in algebra, including collecting like terms, and expanding and factorising using the distributive law. The resource emphasises the importance of algebra in generalising and justifying arithmetic results. It is assumed that students have some familiarity with algebraic notation.

Lesson 1: Addition Chain

This resource starts with a teacher-led calculation short cut and asks students to think about how it is done and why it works. Students use spreadsheets to search for similar relationships and use algebra to explain which of these are always true. The task can be used as a springboard for an exploration of the Fibonacci sequence.

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

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Reflection on this sequence

Rationale

Approaching algebra as generalised arithmetic shows students the power of algebra when abstracting number. This focus on algebra as generalised arithmetic is typically under-represented in secondary mathematics in favour of more time spent on functions and equations.

reSolve mathematics is purposeful

- This sequence supports a rich interpretation and enactment of the Australian Curriculum: Mathematics, providing fun and engaging ways to understand the algebraic content of the Curriculum. The lesson draws on well-established mathematical concepts to identify the algebra within the context of numerical sequences; it also explores well-known sequences such as the Fibonacci sequence.

reSolve tasks are inclusive and challenging

- The task activates existing knowledge, develops new knowledge and explores relationships between key ideas in the Australian Curriculum. Students are required to navigate a variety of different statistical and mathematical software in ways they are unlikely to be experienced with, and which include a wide variation of prompts and programs to suit students.

reSolve classrooms have a knowledge-building culture

- The task in this sequence begins by inspiring curiosity and intrigue through a shared classroom experience that promotes higher-order thinking through the role of both teacher and student. Students build understanding through collaborative inquiry, action and reflection. It encourages students to challenge their existing concepts and to use their mistakes as a vehicle for further learning.

reSolve MATHS BY INQUIRY Addition Chain

Addition Chain

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About this lesson

This resource prompts students to examine and generalise numerical relationships arising from 'addition chains'. These are recursive sequences, such as the Fibonacci sequence, generated by summing the previous two terms. The teacher demonstrates a calculation short cut to create a sense of curiosity about how it is done and why it works. This leads to a student search for similar relationships, using spreadsheets. The resource requires students to use algebra to justify the conditions for which these relationships hold true.

Note: This resource uses algebra to prove generalisations made in arithmetic. Hence, although the algebraic operations involve no more than collecting like terms and identifying common numerical factors in linear expressions, the level of reasoning involved is sophisticated.

Australian Curriculum: Mathematics (Year 8)

ACMNA191: Factorise algebraic expressions by identifying numerical factors.

ACMNA192: Simplify algebraic expressions involving the four operations.

Mathematical purpose

- This resource focuses on mathematical reasoning using algebra. Students use spreadsheets to investigate potential arithmetic relationships, then use algebra to identify and justify which relationships are generally true. The task can be used as a springboard for an in-depth exploration of the Fibonacci sequence and develops skills in using spreadsheets.

Learning intention

- To design addition chains and explore their generalised properties using algebraic notation.

Time

Two to three lessons of approximately 1 hour each.

Resources

- reSolve Excel Spreadsheet *1a Addition Chain* (optional)

Vocabulary

- Fibonacci sequence

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Demonstrating a mathematical magic trick

Set the scene by reminding students of other situations in which they have created a mathematical 'magic trick' (e.g. in the reSolve Year 8 resources Think of a Number and Tens and Units) or by asking them to share their own examples of mathematical 'magic tricks'.

Pose the challenge: *All these so-called magic tricks have a basis in logic — they can be explained! Our challenge is to go beyond the entertaining to understand and invent our own magic tricks. We are going to look at one example, understand why it works and see if we can use our understanding to invent a similar one.*

Ask each student to:

- Write down two numbers, vertically aligned. (These are their starting numbers or 'seed numbers'.)
- Add them.
- Write the answer under the second number.
- Add the last two numbers in the list.
- Repeat until there are 10 numbers in the list.
- Add the total of all 10 numbers in the list. In the example shown at right, the total will be 825.

7
5
12
17
29
...
...

Ask a student to come to the board, record their seed numbers and begin to write down their full list. As the student writes the 10th number, quickly tell them the total of their numbers. Make it look as though you have added all 10 numbers very quickly. In fact, you have **multiplied the seventh number by 11** while the student has been writing the remaining numbers.

Teacher notes:

- To multiply the seventh number by 11, start by writing the final digit of the number on the right-hand end, and then successively add pairs of digits from the right, carrying if necessary. End with the left-hand digit or one more than the digit if carrying has been necessary.
  - ◊ For example:
    - ▪ To multiply 312 by 11, the digits in the answer, starting from the right, will be 2, $2 + 1 = 3$, $1 + 3 = 4$, (3). Therefore, $312 \times 11 = 3432$.
    - ▪ To multiply 467 by 11, the digits will be 7, $7 + 6 = 13$ (3), $6 + 4 = 10$ with 1 carried over makes 11 (1), 4 with 1 carried over makes 5. Therefore, $467 \times 11 = 5137$.
    - ▪ To multiply 876 by 11, the digits are 6, $6 + 7 = 13$ (3), $7 + 8 = 15$ with 1 carried over makes 16 (6), 8 with 1 carried over makes 9. Hence, $876 \times 11 = 9636$.
- With practice, it is possible to work from the left, anticipating the carries.

Ask a second student to come to the board and write their full list but stop after the eighth number (for example). Again, tell them the total of all 10 numbers.

This poses the challenge of how it was possible to find the total so quickly, in fact without even seeing the last two numbers.

The relationships of numbers in addition chains

Give students time to look at their list of numbers and at their total, to see if they can work out how it is possible to find the total without actually adding all their numbers.

Ideally, someone will develop a hypothesis that the total is 11 × the seventh number.

Enabling prompt:

- *Subtract each of the numbers in your list from the total.*
  Using the original example, the results will be:
  - ◊ 825 – 7 = 818
  - ◊ 825 – 5 = 820
  - ◊ 825 – 12 = 813
  - ◊ 825 – 17 = 808
  - ◊ 825 – 29 = 796
  - ◊ 825 – 46 = 779
  - ◊ **825 – 75 = 750**
  - ◊ 825 – 121 = 704
  - ◊ ...
- The seventh subtraction always provides an interesting result: the answer is 10 times the number being subtracted. What does this mean about the relationship between the seventh number and the total of the 10 numbers?
  - ◊ Total sum – $T_7 = 10T_7$; hence, total sum $= 11T_7$.

Checking that the relationship can be generalised

Ask a few students to come to the board and record only the seed numbers and the seventh number. Do not record the total at this stage.

Here is a sample board.

Name of student	David/Jemma	Ahmed	Yujin	Karen/Truong
First number	4	3	8	6
Second number	7	2	10	2
...	...	...	...	...
Seventh number	76	31	120	46
Total of 10 numbers				

Complete the total for some of the columns and ask the students to verify the total to the rest of the class; for example:

- 'David and Jemma, your total is 836. Is that correct?'
- 'Ahmed, your total is 341. Is that correct?'

Ask the class to predict the total for the other students listed on the board and ask those students to verify the results.

Teacher note:

- If students have completed the reSolve Year 8 sequence Tens and Units, they may be able to multiply by 11 quickly. If not, they can find the answer quickly by multiplying by 10 and then adding the original number.
  For example: $47 \times 11 = 470 + 47 = 517$

Investigating other relationships

Ask students what happens when we go beyond 10 numbers. Is there a quick way to calculate the sum of the first 11 numbers without actually doing the addition? Or the first 12 numbers? Or …?

Teacher notes:

- Although this can be done by hand, it is much easier to get students to set up a spreadsheet, input their own seed numbers and then fill down beyond 10 numbers (see Table 1).

Table 1

	A	B	C
1		**Addition chain**	**Factor of sum**
2	**Seed *a***	7	=B$13/B2
3	**Seed *b***	5	=B$13/B3
4		=B2+B3	=B$13/B4
5		=B3+B4	=B$13/B5
…		…	…
12		=B10+B11	=B$13/B12
13	**Sum**	=SUM(B2:B12)	

◊ Cell B4 contains a formula for adding the two previous numbers, which can then be filled down as far as desired. Cell B13 contains a formula for the sum of the numbers in the chain.

◊ Column C checks to see if each of the numbers in the chain is an integer factor of the sum. The $ sign before the 13 is an absolute cell reference that ensures that when the formula is filled down, the number in row (the sum) is used in every formula.

- **Resources:** This spreadsheet is supplied as the reSolve Excel Spreadsheet *1a Addition Chain* and uses the formulas shown in Table 1.
- To create an addition chain of length 12, it will be necessary to insert a row before row 13, ensure that the formula sums the numbers B2 to B13 and that the reference in column C is to B$14.
- When using the seed numbers 7 and 5 and filling down to sum 11 numbers, no integer factors appear. The same is true when 12 numbers are summed. The first integer factor that appears is when 14 numbers are summed, which is when the sum is 29 times the ninth number in the chain. This relationship holds true regardless of the seed numbers used.

Suggest that these are not the only relationships between numbers in the addition chain and the total of some of the numbers. We will explore this using a longer list and keeping a cumulative total.

Start a list on the board, similar to Table 2 below. (Note that column letters and row numbers are given here for internal reference only.)

Table 2

	A	B
1	**Addition chain**	**Cumulative sum**
2	7	7
3	5	12
4	12	24
5	17	41
6	29	70
7	46	116

Ask students to find other relationships in this table that *appear* to be true.

For example, in Table 2, the third number in the Cumulative sum column [B(4)] appears to be twice that of the third number in the Addition chain column [A(4)]. Or B(6) = 14A(3). Or B(3) = 1A(4). However, these may not be true for other pairs of seed numbers.

Ask students to generate some possible relationships to investigate using their own pair of seed numbers.

T Teacher notes:

- Again, we recommend that students set up a spreadsheet and input their own seed numbers. Rows 2 and 3 in column A of Table 3 contain two seed numbers. Cell B4 contains the formula to generate the addition chain, as done previously. Column C gives one formula for the cumulative sums.

  Table 3

	A	B	C
1		**Addition chain**	**Cumulative sum**
2	**Seed *a***	7	=B2
3	**Seed *b***	5	=C2+B3
4		=B2+B3	=C3+B4

- This spreadsheet is supplied as the reSolve Excel Spreadsheet *1a Addition Chain* and uses the formulas shown in Table 3.

Some of the relationships students will find are always true, and some are true only for special cases.

Make a list on the board of possible relationships. As a class, refine the list by crossing out relationships that are true only in special cases (i.e. students identify and remove relationships that do not apply to their results).

Explain to students that one exception (as long as the arithmetic is correct!) suffices to show that a particular relationship is not generally true. However, even though we have several verifications of other relationships, it might be that all the pairs of seed numbers we have chosen happen to satisfy them but other seed numbers will not. To prove that a relationship is *always* true, we need to use algebra to generalise.

An algebraic explanation and exploration

Begin to create a generalised Addition chain and Cumulative sum table, as shown in Table 4.

Table 4

	A	B
1	**Addition chain**	**Cumulative sum**
2	a	a
3	b	$a + b$
4	$a + b$	$2a + 2b$
5	$a + 2b$	$3a + 4b$
6	$2a + 3b$	$5a + 7b$
7	$3a + 5b$	$8a + 12b$

Ask students to use the generalised results to write down some relationships that must always be true. For example, Table 4 shows that cell B(3) = 1A(4) and that cell B(7) = 4A(6), as in Table 2. But in Table 2, cell B(6) = 14A(3), which does not hold true in Table 4.

Justify the relationships by factorising $8a + 12b$ as $4(2a + 3b)$.

T Teacher notes:

- Again, although this can be done by hand, it is much easier for students to set up a spreadsheet. Table 5 below is part of the reSolve Excel Spreadsheet *1a Addition Chain* and uses the formulas given previously.
- Note that this spreadsheet also has a column labelled ***n*** to help keep track of which terms are being considered in the sequence or series.
- It is, of course, preferable to ask students to make their own spreadsheet.

Table 5

	A	B	C	D	E
1		**Addition chain**		**Cumulative sum**	
2	***n***	**# of *a***	**# of *b***	**# of *a***	**# of *b***
3	**1**	1	0	=B3	=C3
4	**=A3+1**	0	1	=D3+B3	=E3+C3
5	**=A4+1**	=B3+B4	=C3+C4	=D4+B4	=E4+C4
6	**=A5+1**	=B4+B5	=C4+C5	=D5+B5	=E5+C5

Ask students to use their spreadsheet to find other relationships that must always be true.

Table 4 shows that the first two relationships that are always true are B(3) = 1A(4) and B(7) = 4A(6). The next relationship that is always true is B(11) = 11A(8). This is the relationship used in the original 'trick' at the start of the lesson: that the total of the first 10 numbers in the addition chain is equal to 11 × the seventh number.

Possible student response:

- The first six relationships, excluding $B(2) = A(2)$ and $B(4) = 2A(4)$, are:

$a + b = 1(a + b)$	or $B(3) = 1A(4)$
$8a + 12b = 4(2a + 3b)$	or $B(7) = 4A(6)$
$55a + 88b = 11(5a + 8b)$	or $B(11) = 11A(8)$
$377a + 609b = 29(13a + 21b)$	or $B(15) = 29A(10)$
$2584a + 4180b = 76(34a + 55b)$	or $B(19) = 76A(12)$
$17\,711a + 28\,656b = 199(89a + 144b)$	or $B(23) = 199A(14)$

Note: The relationships for B(2) and B(4), although true, have been excluded, as they do not fit the pattern.

Teacher note:

- If students find these relationships interesting, they might like to investigate the sequence of ratios 1, 4, 11, 29, 76, 199 by typing into the Online Encyclopaedia of Integer Sequences at https://oeis.org/. The sequence of numbers was contributed by Lekraj Beedassy on 31 December, 2002. There is also a list of the first 200 numbers in the sequence at https://oeis.org/A002878/b002878.txt.

Further activities

Activity 1

Ask students to explain how they could use the relationship $B(23) = 199A(14)$ as the basis of another addition 'trick'.

Activity 2

Use other seed numbers to investigate what happens in the addition chain. For example:

- What happens when one of the seed numbers is zero?
- What happens when one or both seed numbers are negative? Will it always be the case that all the terms will eventually become all positive or all negative?

Activity 3

Find numbers for which the relationships that are not generally true will work.

For example, if $B(6) = 14A(3)$, then $5a + 7b = 14b$. Hence, $5a = 7b$, where $a = 7$and $b = 5$ is one possible pair of numbers for which this will be true. All others are multiples of $a = 7$ and $b = 5$.

Choose a relationship that you would 'like' to work and find a pair of seed numbers for which it will work. For example, if we would like $B(8) = 12A(5)$ to work, then $13a + 20b = 12(a + 2b)$, where $a = 4b$. Therefore, we could use $a = 4$ and $b = 1$ as the seed numbers. Students can confirm that the relationship works using the spreadsheet.

Note that this is an introduction to simple Diophantine equations.

Activity 4

Students could explore starting with three numbers and adding the previous three terms (i.e. a third-order recursive relationship). For example:

Table 6

	A	B
1	**Addition chain**	**Cumulative sum**
2	7	7
3	5	12
4	1	13
5	13	26
6	19	45
7	33	78
8	65	143
9	117	260
10	215	475

Although we can observe several relationships between the numbers in Table 6, the only relationship that holds generally is that B(9) = 4A(8).

Activity 5

Students could explore properties of Fibonacci numbers using 1 and 1 as seed numbers. Some possibilities for investigation include:

- Explain why the pattern of odd and even numbers is O, O, E, O, O, E, ...
- Investigate and prove a pattern involving multiples of 3.
- Explain why the sum of three consecutive Fibonacci numbers is twice the largest number in the sum.
- Find out about the history of the Fibonacci numbers and their link to the golden ratio. A useful link is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html.

A good source of Fibonacci puzzles is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibpuzzles.html.

Students could also explore Lucas numbers, the sequence commencing 2, 1, 3, 4, 7, 11, 18, 29, ... It is interesting to note that every second Lucas number (1, 4, 11, 29, ...) corresponds to the ratios found in the exploration of the relationships between the cumulative sums and the numbers in the addition chain. Students might also spot the fact that for the sequence 1, 4, 11, 29, ..., a_{n+1}, the general rule is $3a_n - a_{n-1}$. A good reference for Lucas numbers is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/lucasNbs.html.

A Fibonacci-like sequence in which any two numbers can be used as seed numbers, such as those used in the activity, is called a G series or a generalised Fibonacci sequence. A good reference for the G sequence and its properties is http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibGen.html.