re(Solve) MATHS BY INQUIRY Bent Bamboo

Y10

Summary of learning goals

- Students apply Pythagoras' theorem to explore a historical real-world problem. They compare their modern solutions with the historical solution.

Australian Curriculum: Mathematics (Year 10)

ACMNA233: Expand binomial products and factorise monic quadratic expressions using a variety of strategies.

ACMNA239: Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials, using digital technology as appropriate.

Summary of lessons

Who is this sequence for?

- This lesson assumes knowledge of Pythagoras' theorem and makes links to the algebra of binomial expansions and to graphs of quadratic functions.

Lesson 1: Bent Bamboo

Students use Pythagoras' theorem to solve a problem from an ancient Chinese text. They make physical models of the problem and use this to construct a graph. They use algebra skills associated with binomial expansions and use simplification of fractions to show that the general solution given in the Chinese text gives identical results to those developed using modern mathematical techniques. There are opportunities to solve similar problems for consolidation.

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

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Reflection on this sequence

Rationale

The concepts behind Pythagoras' theorem and its proofs are taught well in classrooms, and there are ample high-quality resources to assist teachers. For this reason, this lesson is not intended to prove or teach the theorem. However, traditional applications are often shallow and uninteresting and so the focus here is to provide explorative and atypical applications.

This task provides a historical look at the appearance of Pythagoras' theorem in a different culture and represented in a different way.

reSolve mathematics is purposeful

- This sequence provides an interesting application of Pythagoras' theorem that is aimed at building students' fluency with calculations in unorthodox contexts.

reSolve tasks are inclusive and challenging

- The task can be solved via practical experimentation or calculation.

reSolve classrooms have a knowledge-building culture

- The lesson relies on collaborative inquiry, action and reflection to reach a solution.

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Bent Bamboo

Y10

About this lesson

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Mathematical purpose

- Students model a problem from an ancient Chinese mathematical text. They generalise their solution and show that the relationship between the height of the break and the horizontal distance forms part of a parabola. They simplify the algebraic expressions to show that the general solution obtained is identical to that given in the Chinese text.

Learning intention

- To solve an ancient Chinese mathematical text using modern techniques and compare it to the historical solution.

Time

Three lessons of approximately 1 hour each.

Resources

- reSolve PDF *1a Bent Bamboo*
- Pipe cleaners, florist wire (from a $2 shop) or other props that can represent bent pieces of bamboo.

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The *kou-ku* theorem

Resources: Show students the image below, included as reSolve PDF *1a Bent Bamboo*.

This is an ancient Chinese maths problem from the earliest known Chinese mathematical text, the *Chou Pei Suan Ching*, dating from around 500 to 200 BCE. It concerns the *kou-ku* (pronounced go-goo) theorem.

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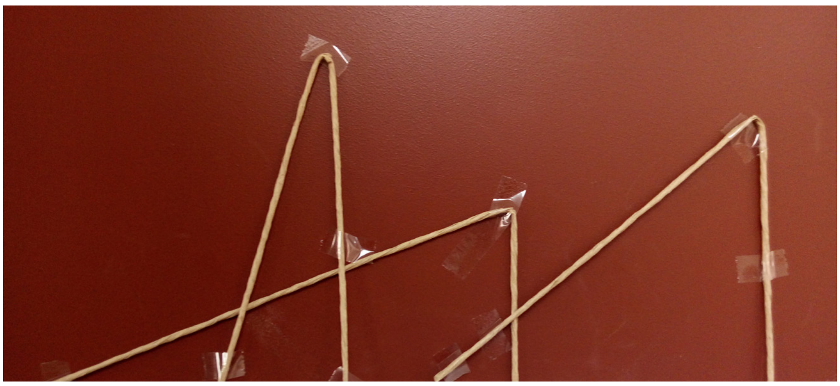
Ask students what they think the problem might be about. They are likely to say that it is about right-angled triangles.

Provide students with some ‘bamboo’ of fixed length that they can break. Ask students to bend the bamboo and arrange the bent bamboo so that the two ends touch the ground and one side is vertical, as shown in the image below.



Have a short class discussion, considering questions such as:

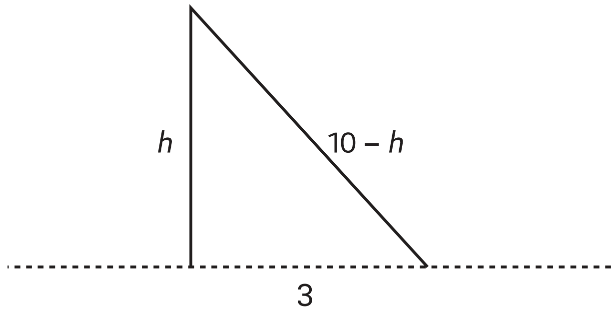
- *If the bamboo is 1 m tall, what are the minimum and maximum lengths of the vertical side? What are the minimum and maximum horizontal distances from the base of the bamboo to the point where it touches the ground?*
- *If you know the height of the break, how would you calculate the horizontal distance from the base of the bamboo to where the top touches the ground?*
- *If you know the horizontal distance from the base of the bamboo to where the top touches the ground, how would you calculate the height of the break?*

Ask students to set their 'bamboo' aside for later in the lesson.

Give students the original question translated into English: *There is a bamboo 10 chi high, the upper end of which, being broken, touches the ground at 3 chi from the foot of the stem. What is the height of the break?* Note: a chi is about 23 cm.

Give students some time to solve the problem using their knowledge of Pythagoras' theorem. Students may use a variety of methods, including:

- Using the bamboo they bent, choosing a suitable scale and measuring the height.
- Trial and improvement by choosing a value for height and then calculating the horizontal distance to see if it is 3 chi. This could be done with a spreadsheet.
- Formulating the problem algebraically and solving the resulting equation:



$$3^2 + h^2 = (10 - h)^2$$

$$9 + h^2 = 100 - 20h + h^2$$

$$9 = 100 - 20h$$

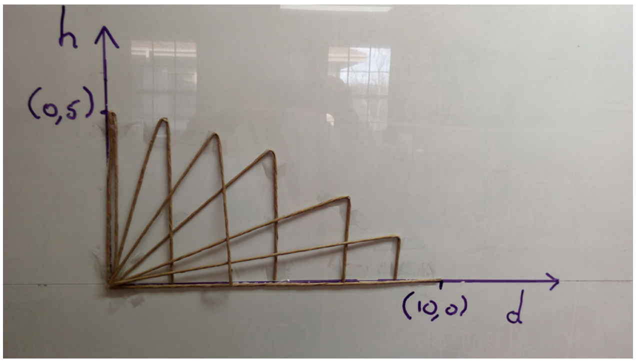
$$h = \frac{91}{20}$$

Enabling prompts:

- Ask students what the length of the hypotenuse of the triangle would be if the break is at 2 chi. Can they generalise their findings?
- When $h = 0$, what is the horizontal distance? When the horizontal distance is zero, what is the height?

Exploring the relationship

In this part of the lesson students compare their pieces of bamboo to get a sense of the relationship between the height of the break and the horizontal distance from the base of the bamboo to where the top touches the ground.



Draw a pair of axes on the whiteboard to represent the horizontal distance (d-axis) and height of the break (h-axis). Ask students to orient their bent bamboo so that the vertical side is to the right. Now stick it to the axes so that the point where it touches the ground is at the origin and the vertical section is parallel to the h-axis. The height of the break then forms part of an inverted parabola with vertex at (0, 5) and d-intercept at (10, 0) (measured in chi). Ask students what is the physical significance of the vertex and the d-intercept.

Ask students to find the equation of the parabola.

They may do this using the vertex and y-intercept, writing it as $h = ad^2 + 5$. Then, using $d = 10, h = 0$ because one point on the curve gives $a = -0.05$. Hence, $h = -0.05d^2 + 5$.

Alternatively, using Pythagoras' theorem:

$$d^2 + h^2 = (10 - h)^2$$

$$d^2 + h^2 = 100 - 20h + h^2$$

$$d^2 = 100 - 20h$$

$$h = \frac{-d^2}{20} + 5$$

Note that these solutions are the same.

Reflection

Provide students with the solution method shown in the Chou Pei Suan Ching:

> *Take the square of the distance from the foot of the bamboo to the point at which its top touches the ground, and divide this by the length of the bamboo. Subtract the result from the length of the bamboo, and halve the resulting difference. This gives the height of the break.*

Ask students to formulate this algebraically, using l as the length of the bamboo, d as the horizontal distance from the base of the bamboo to where the top touches the ground, and h as the height of the break.

The algebraic formulation is $h = \frac{1}{2}\left(l - \frac{d^2}{l}\right)$.

Ask students to simplify this and compare it with the solution obtained using Pythagoras' theorem and algebra.

$$h = \frac{1}{2}\left(l - \frac{d^2}{l}\right)$$

$$h = \frac{l^2 - d^2}{2l}$$

Using Pythagoras and algebra, we obtain:

$$h^2 + d^2 = (l - h)^2$$

$$h^2 + d^2 = l^2 - 2lh + h^2$$

$$2lh = l^2 - d^2$$

$$h = \frac{l^2 - d^2}{2l}$$

The general solution given in the Chinese text shows that the *kou-ku* theorem was known before it was stated and proved by Greek mathematicians such as Pythagoras. Several related problems were given in the Chinese texts, as well as a proof using the principle of 'piling up of rectangles' (see Activity 3 below).

Further activities

Activity 1: Vine around a tree

A problem from Chapter 9 of *Jiu Zhang*, a mathematical text written in 1247 by Chinese Southern Song dynasty mathematician Qin Jiushao, states:

> *Under a tree 20 chi high and 3 chi in circumference, there grows an arrowroot vine that winds seven times round the stem of the tree and just reaches its top. How long is the vine?*

Solve the problem.

Enabling prompt:

- Use a cylindrical can with a label to model the tree. Draw a line around the label that winds around it exactly once and touches the top of the can at a point directly above where it starts at the bottom of the can. Cut the label vertically through the two ends of the line. What shape do you make and how would you calculate the length of the line?

Activity 2: Rope hanging from a tree

Another problem from Chapter 9 of *Jiu Zhang* states:

> *There is a rope hanging from the top of a tree with 3 chi of it lying on the ground. When it is tightly stretched, so that its end just touches the ground, it reaches a point 8 chi from the base of the tree. How long is the rope?*

Solve the problem.

Enabling prompts:

- If the tree is 7 chi tall, how long is the rope? How far does the rope reach from the base of the tree?
- Draw diagrams, carefully choose a variable to represent one of the unknown lengths, and write other lengths in terms of the variable you chose.

Compare your solution with the method given in the *Jiu Zhang*:

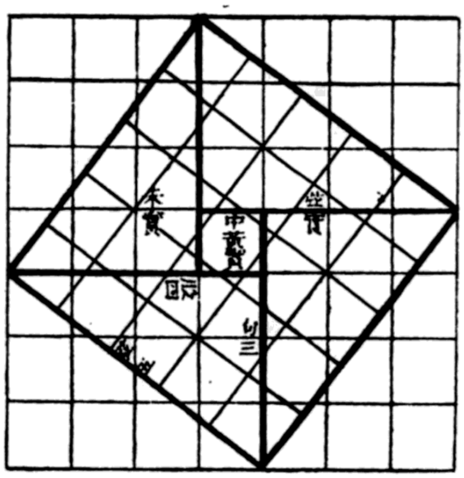
> *Divide the square of the distance between the base of the tree and the end of the rope by the length of rope lying on the ground. Add to this the length of rope lying on the ground and halve the result. This gives the length of the rope (given as $12\frac{1}{6}$ chi).*

Activity 3: Piling up the rectangles

Chinese texts gave the following proof of the *kou-ku* theorem:

> *Let us cut a rectangle (diagonally), and make the width 3 (units) wide, and the length 4 (units) long. The diagonal between the (two) corners will then be 5 (units) long. Now, after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles, of width 3, length 4 and diagonal 5, together make two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder is of area 25. This (process) is called 'piling up the rectangles'.*

The text was accompanied by this diagram:



Draw a similar diagram and pile up the rectangles for a right-angled triangle with sides 5, 12 and 13.

An extension of this proof to a general case was achieved in different ways by Zhao Zhujing and Liu Hui, two Chinese commentators living in the third century AD. Their method is essentially that shown in the Vi Hart Youtube clip at https://www.youtube.com/watch?v=z6lL83wl31E.

Two other proofs using algebra are given in the Further activities section of Lesson 1: Quarter Squares of the reSolve Year 10 resource Sums and Differences of Squares.