re(Solve) MATHS BY INQUIRY

Rectilinear Polygons

Y9

Summary of learning goals

- Students use geometric reasoning to establish relationships between angles in particular types of polygons.

Australian Curriculum: Mathematics (Year 9)

ACMMG202: Establish properties of quadrilaterals using congruent triangles and angle properties, and solve related numerical problems using reasoning.

ACMNA213: Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate.

ACMNA215: Sketch linear graphs using the coordinates of two points and solve linear equations.

Summary of lessons

Who is this sequence for?

- This sequence assumes that students have a sound understanding of the types and measure of angles. Knowledge of the angle sum of a polygon is helpful or can be developed as needed during the first lesson. In the first lesson, elementary algebra is helpful to prove the results relating to 90°-270° polygons.

Lesson 1: 90°-270° Polygons

Students draw polygons that contain only 90° and 270° angles. They establish a relationship between the number of 90° and 270° angles, and prove their result using the angle sum of a polygon. They generalise the result to any number of sides.

Lesson 2: Arranging Angles

Students look at how many different ways they can arrange the angles in a 90°-270° decagon. They explain their reasoning using the number of different ways that they can partition numbers into two parts. The lesson offers opportunities for students to be creative in how they arrange the angles and has the potential for students to use coding to construct rectilinear polygons.

Lesson 3: Maximum Number of 90°-270° Angles

Students construct polygons containing a large number of 90° angles or a large number of 270° angles. They make generalisations about the maximum number of 90° or 270° angles in any given polygon, and justify their generalisations using the angle sum of a polygon. There are opportunities to construct polygons using the coding program Scratch.

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Reflection on this sequence

Rationale

This sequence focuses on developing mathematical reasoning through the investigation of angles in a particular class of polygons. It links geometry, algebra and combinatorics through the generalisation and justification of a geometric result and through consideration of the number of different possible arrangements of angles.

reSolve mathematics is purposeful

- The sequence shows the interconnectedness of mathematical concepts, and illustrates how ideas from one strand of mathematics can be used to investigate and generalise results in another strand.
- The opportunity to use coding to construct polygons links to the Digital Technologies component of the Australian Curriculum.

reSolve tasks are inclusive and challenging

- The initial task of constructing 90°-270° polygons is low floor and high ceiling in that it commences with students drawing polygons that they will have seen before, but concludes by generalising to find the number of 90° and 270° angles in rectilinear polygons with any even number of sides.
- Students engage in sustained inquiry, guided by teacher questioning.

reSolve classrooms have a knowledge-building culture

- Students build success and understanding by initially working individually to construct a 90°-270° polygon, then pooling results to generate a collection of such polygons from which they can look for patterns and make generalisations.
- The lessons are likely to challenge students' perceptions of what polygons look like.

LESSON 1

re(Solve) MATHS BY INQUIRY Rectilinear Polygons

90°-270° Polygons

Y9

About this lesson

Students draw polygons that contain only 90° and 270° angles (here called 90°-270° polygons). They establish a relationship between the number of 90° and 270° angles, and prove their result using the angle sum of a polygon. They generalise the result to any number of sides.

Australian Curriculum: Mathematics (Year 9)

ACMMG202: Establish properties of quadrilaterals using congruent triangles and angle properties, and solve related numerical problems using reasoning.

ACMNA213: Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate.

Mathematical purpose

- To explore properties of polygons that contain only 90° and 270° angles and to use algebra to justify findings.

Learning intention

- To learn about polygons that contain only 90° and 270° angles.

Time

A lesson of approximately 1 hour.

Vocabulary

- angle sum of a polygon
- polygons

Resources

- GeoGebra worksheet Polygon Angle Sum at https://www.geogebra.org/m/GBHsT33x (if students are unfamiliar with the formula for the angle sum of a polygon).
- protractors
- rulers

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90° and 270° angles

Pose the question: *Draw a polygon that consists of only 90° and 270° angles.*

Explain that these will be referred as 90°-270° polygons. Each student draws a 90°-270° polygon.

Have students find someone who has drawn a polygon with a different number of sides to their own polygon. Ask the pairs of students to draw a third 90°-270° polygon but with a different number of sides to the two they have already drawn.

Each pair of students will now have at least three different polygons; for example:

90°-270° polygons

Discuss the similarities and differences between the polygons and any patterns that students notice.

Teacher notes:

- The important properties for students to recognise are:
  - ◊ the number of sides of each polygon
  - ◊ the number of 90° angles in each polygon
  - ◊ the number of 270° angles in each polygon.
- Do **not** tell the students what to look for.

As a class, collate students' findings. Students should observe that:

- All 90°-270° polygons have an even number of sides.
- Each polygon will *always* contain four more 90° angles than 270° angles.

Generalising the geometry

Ask students how many 90° angles would be in a 90°-270° polygon with one hundred sides.

T **Teacher notes:**

1. There are 100 angles in a 100-sided polygon.
2. If the number of 90° angles is n, then the number of 270° angles is $n-4$. So, $n+(n-4)=100$.
3. Hence, $n=52$. The polygon will contain 52 90° angles and 48 270° angles.

Suggest that for a polygon with an even number of sides, the number of sides can be conveniently written as $2s$. Ask students to generalise the number of 90° angles and the number of 270° angles in a 90°-270° polygon with $2s$ sides. There are $s+2$ 90° angles and $s-2$ 270° angles.

Ask students to show that for a polygon with $2s$ sides, $s+2$ 90° angles and $s-2$ 270° angles result in an angle sum of $180(2s-2)°$.

T **Teacher notes:**

- The angle sum of a polygon, in degrees is, $180(n-2)$, where n is the number of sides.
- When $n=2s$, the angle sum $=180(2s-2)°$.
- Now, $90(s+2)+270(s-2)=90s+180+270s-540=360s-360=180(2s-2)°$, as required.

Ask students to choose an even number of sides, calculate the number of 90° and 270° angles, and see if they can draw the corresponding 90°-270° polygon.

LESSON 2

re(Solve) MATHS BY INQUIRY

Rectilinear Polygons

Arranging Angles

Y9

About this lesson

Students look at how many different ways they can arrange the angles in a 90°-270° decagon. They explain their reasoning using the number of different ways that they can partition numbers into two parts. The lesson offers opportunities for students to be creative in how they arrange the angles and has the potential for students to use coding to construct rectilinear polygons.

Australian Curriculum: Mathematics (Year 9)

- Students reason to construct and communicate mathematical arguments.

Mathematical purpose

- To show the strong links between geometry, arithmetic and combinatorics.

Learning intention

- How many different 90°-270° decagons can you make? How do we know if we have made all the possible decagons?

Time

A lesson of approximately 1 hour.

Vocabulary

- decagon

Resources

- GeoGebra worksheet Polygon Angle Sum at https://www.geogebra.org/m/GBHsT33x (if students are unfamiliar with the formula for the angle sum of a polygon).
- 90-270 polygons Scratch program (see https://scratch.mit.edu/projects/222231380/)
- reSolve *2a Shine Dome Polygon* image

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Teacher background information

In this lesson we create a categorisation system for 90°-270° decagons that identifies the location of each 90° and 270° angle. Note that in a 90°-270° decagon there are seven 90° angles and three 270° angles, as found in Lesson 1.

Consider the following three decagons, for which the sequence of angles starts at the top left-hand corner and then moves in a clockwise direction.

Decagon 1

The sequence of angles is: 90°, 90°, 270°, 90°, 270°, 90°, 270°, 90°, 90°, 90°. If we start recording from the first 270°, the sequence is: 270°, 90°, 270°, 90°, 270°, 90°, 90°, 90°, 90°, 90°.

We can write this sequence as 1, 1, 5 to represent the number of 90° angles separated by each 270° angle. Each 270° angle represents a break. Since there are three 270° angles, the number of right angles is partitioned into three parts. We will call the 1, 1, 5 sequence the **right angle sequence**.

Note: If we start our sequence with a different 270° angle, it might be written as 1, 5, 1 or 5, 1, 1. These shapes are rotations or reflections of each other (when ignoring differences in side lengths). This means that the right angle sequence can be written in any order without changing the shape. We will write them with the smallest number first, which will be useful later in the investigation when students find all possible 90°-270° decagons.

Decagon 2

The sequence of angles is: 90°, 90°, 270°, 90°, 90°, 90°, 90°, 270°, 270°, 90°. If we start recording from the first 270° angle, the sequence is: 270°, 90°, 90°, 90°, 90°, 270°, 270°, 90°, 90°, 90°

The right angle sequence of this decagon is 4, 0, 3, where the 0 represents two 270° angles together. Starting with the smallest number in the sequence, we will write it as 0, 3, 4.

Decagon 3

The sequence of angles is: 90°, 90°, 90°, 90°, 270°, 90°, 270°, 90°, 270°, 90°. If we start recording from the first 270° angle, the sequence is: 270°, 90°, 270°, 90°, 270°, 90°, 90°, 90°, 90°, 90°.

The right angle sequence of this decagon is 1, 1, 5. In terms of their right angle sequences, decagon 1 and decagon 3 are the *same*. Geometrically, decagon 3 could be transformed into decagon 1 by rotation and stretching of sides. However, decagon 2 cannot be transformed into decagon 1.

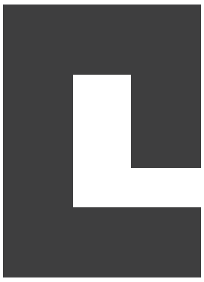
Drawing decagons

Ask students to draw a 90°-270° decagon. Have students compare their decagon to someone else's decagon and to consider the similarities and differences.

Introduce the students to the categorisation system outlined in the Teacher background information and ask them to categorise their decagon.

Pose the challenge: *Can you draw a decagon with all the right angles next to each other? What would be its right angle sequence?*

Expected student response:



Students will find that a decagon with all the right angles next to each other forms a spiral.

Starting from the first 270° angle, the angle sequence is:
270°, 270°, 270°, 90°, 90°, 90°, 90°, 90°, 90°, 90°.

This means the right angle sequence is 0, 0, 7.

Resources: Students might wish to experiment with this Scratch program, which generates a 90°-270° polygon with up to a total of 32 angles and with all the right angles next to each other.

Ask: *What would a 100-sided polygon look like if all the right angles were next to each other?*

All possible 90°-270° decagons

Pose the challenge: *Can you find all the other 90°-270° decagons?*

Discuss with students:

- From Lesson 1 we know that any 90°-270° decagon will have seven 90° angles and three 270° angles.
- The number of partitions in the right angle sequence will always equal the number of 270° angles.
- Therefore, to find all 90°-270° decagons, we need to find all three-part partitions of seven. These partitions will include one or more zeros (to represent adjacent 270° angles).

Students will need to find the partitions and then draw the shapes.

Teacher notes:

- In the Teacher background information we found the partitions 1, 1, 5 and 0, 3, 4, and in the section Drawing decagons we found 0, 0, 7.
- There are five more partitions:
  - ◊ 0, 1, 6
  - ◊ 0, 2, 5
  - ◊ 1, 2, 4
  - ◊ 1, 3, 3
  - ◊ 2, 2, 3

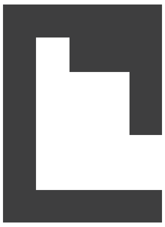
0, 1, 6 0, 2, 5 1, 2, 4 1, 3, 3 2, 2, 3

Enabling prompt:

- *Can you find all the 90°-270°* ***octagons****?*
  - ◊ A 90°-270° octagon will have six 90° angles and two 270° angles. Students will need to find all two-part partitions of six:
    - ▪ 0, 6
    - ▪ 1, 5
    - ▪ 2, 4
    - ▪ 3, 3
  - ◊ Ask students to draw these shapes.

Extending prompt:

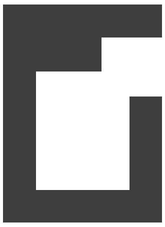
- *Can you find all the 90°-270°* ***dodecagons****?*
  - ◊ A 90°-270° dodecagon will have eight 90° angles and four 270° angles. Students will need to find all four-part partitions of eight (that are not reflections or rotations of each other).
  - ◊ Note the complication when working with four 270° angles, which is described below.



7, 1, 0, 0
90°-270° dodecagon

These are different as one cannot be transformed onto another by rotation, reflection and/or stretching one or more sides.

7, 0, 1, 0
90°-270° dodecagon



These shapes are all the same. The first can be rotated 90° clockwise to create the second.
It is a 7, 0, 0, 1 90°-270° dodecagon.
Stretch and reduce some sides and this is the same as the 7, 1, 0, 0 90°-270° dodecagon.

- We think that there are 29 possible four-part partitions of eight that comply with these restrictions:
  - ◊ If all four numbers are the same or there are three numbers the same and one number is different, then there is only one way that they can be arranged.
  - ◊ If there are two pairs of numbers in the partition or if there are two numbers the same and two are different, then there are two arrangements that will create different shapes.
  - ◊ If all the numbers are different, there are three arrangements that will create different shapes.
- The possible partitions are:

Four numbers the same or three the same and one different	Two pairs of numbers or two the same and two different	Four different numbers
2, 2, 2, 2 only	0, 0, 1, 7 and 0, 1, 0, 7	0, 1, 2, 5 and 1, 0, 2, 5 and 1, 2, 0, 5
0, 0, 0, 8 only	0, 0, 2, 6 and 0, 2, 0, 6	0, 1, 3, 4 and 1, 0, 3, 4 and 1, 3, 0, 4
1, 1, 1, 5 only	0, 0, 3, 5 and 0, 3, 0, 5	
	0, 0, 4, 4 and 0, 4, 0, 4	
	0, 1, 1, 6 and 1, 0, 1, 6	
	0, 2, 2, 4 and 2, 0, 2, 4	
	0, 2, 3, 3 and 2, 3, 0, 3	
	1, 1, 2, 4 and 1, 2, 1, 4	
	1, 1, 3, 3 and 1, 3, 1, 3	
	1, 2, 2, 3 and 1, 2, 3, 2	

Reflection

Create some other interesting arrangements of 90° and 270° angles in a polygon with lots of sides; for example, a castle. Write some Scratch code to draw the polygon.

Resources: Included as an example is the reSolve *2a Shine Dome Polygon* image, which is a 90°-270° polygon rendering of the Australian Academy of Science's Shine Dome.

LESSON 3

re(Solve) MATHS BY INQUIRY

Rectilinear Polygons

Maximum Number of 90°-270° Angles

Y9

About this lesson

Students construct polygons containing a large number of 90° angles or a large number of 270° angles. They make generalisations about the maximum number of 90° or 270° angles in any given polygon, and justify their generalisations using the angle sum of a polygon. There are opportunities to construct polygons using Scratch code.

Australian Curriculum: Mathematics (Year 9)

ACMNA215: Sketch linear graphs using the coordinates of two points and solve linear equations.

Mathematical purpose

- To explore the maximum number of 90° or 270° angles possible in a polygon, and to use algebra to explain the results.

Learning intention

- To investigate the maximum number of 90° or 270° angles possible in a polygon.

Time

A lesson of approximately 1 hour.

Vocabulary

- reflex angle

Resources

- GeoGebra worksheet Polygon Angle Sum at https://www.geogebra.org/m/GBHsT33x (if students are unfamiliar with the formula for the angle sum of a polygon).
- 90-270 polygons Scratch program (see https://scratch.mit.edu/projects/222231380/)

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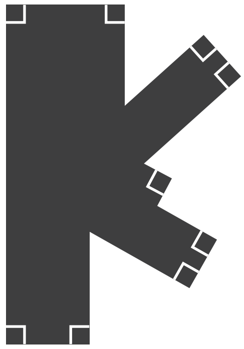
How many 90° angles?

Pose the problem: *Draw a 13-sided polygon that has exactly nine 90° angles.*

Possible student response:



Enabling prompts:

- Draw an 11-sided polygon with eight 90° angles. Can you add two extra sides and an extra right angle?
- Draw a nine-sided polygon with seven 90° angles. Can you add two extra sides and an extra right angle to make an 11-sided polygon with eight 90° angles?

How many 270° angles?

Now ask students to draw a 13-sided polygon that has exactly seven 270° angles.

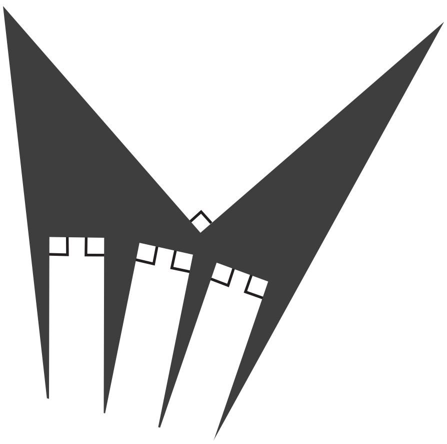
Enabling prompt:

- Allow students to struggle. At an appropriate time remind students of the formula for the angle sum of a polygon and see if they might use this to help. If students are not familiar with this, use the GeoGebra app at https://www.geogebra.org/m/GBHsT33x.

Teacher notes:

- The angle sum of a 13-sided polygon is $180(13-2)° = 1980°$.
- If there are seven 270° angles, this makes a total of $7 \times 270° = 1890°$.
- This leaves 90° remaining to split between the remaining six angles, so they must be very small!

Possible student response:



Finding the maximum number of 90° angles

Ask students to find the maximum number of right angles for a polygon with seven sides.

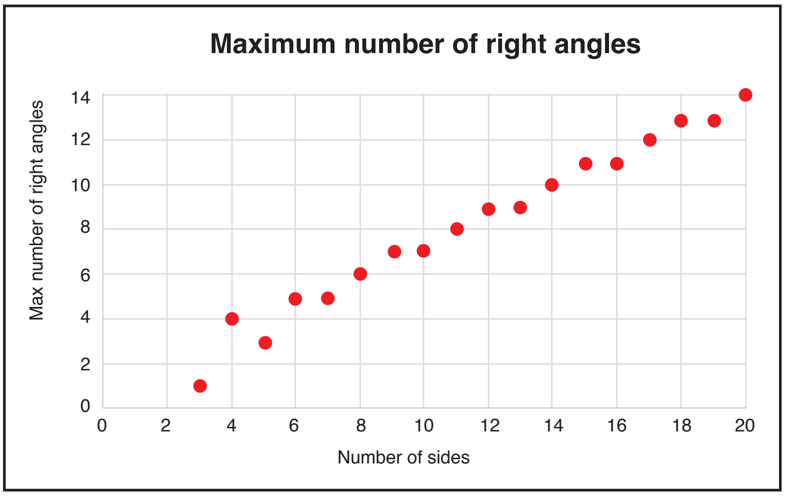
After some trial and error they will be able to make a heptagon with five right angles, but not six. Ask them to justify this finding using the formula for the angle sum of a polygon.

Teacher notes:

- The sum of the angles in a heptagon is $180(7-2)° = 900°$.
- Six right angles would be 540°, leaving 360° for the remaining angle. This is impossible.
- Therefore, the maximum number of right angles is five, which leaves 450° to be shared between the remaining two angles.

Students can complete the table below and graph their results. Discuss what the graph shows.

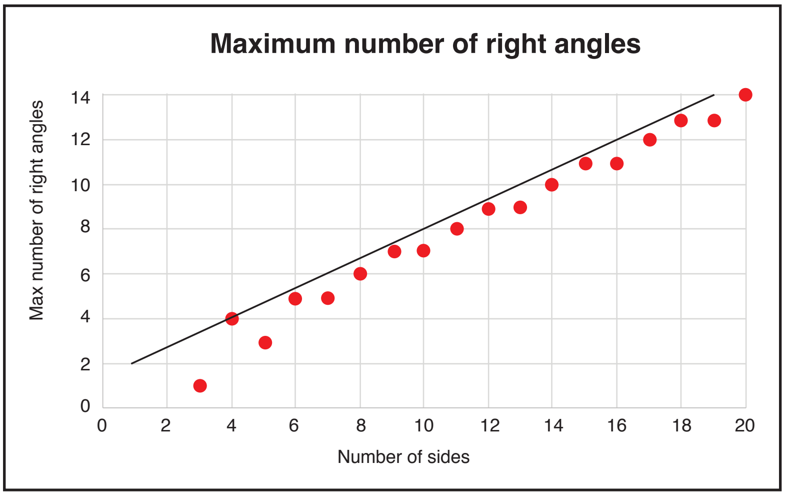
Number of sides	Maximum number of right angles
3	1
4	4
5	3
6	5
7	5
8	6
9	7
10	7
11	8
12	9
13	9
14	10
15	11
16	11
17	12
18	13
19	13
20	14



Note that students are solving the inequality $\frac{180(n-2)-90r}{n-r} < 360$, where n is the number of sides and r is the number of right angles (i.e. every non-right angle in the polygon must be an angle of less than 360°).

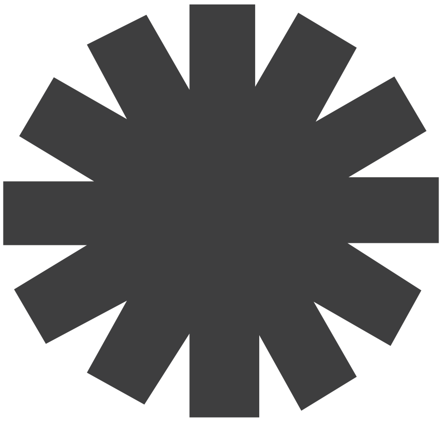
This reduces to $r < \frac{2n+4}{3}$.

The graph on the right includes the line $r = \frac{2n+4}{3}$.

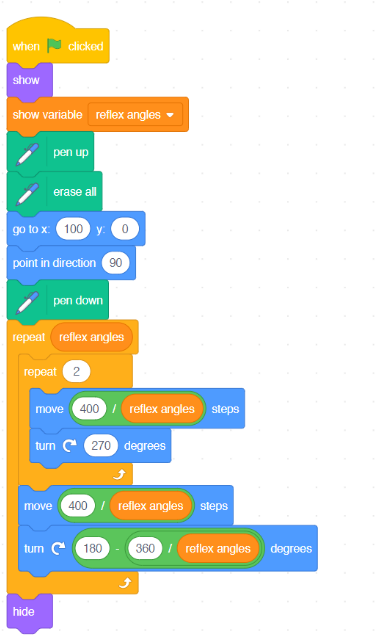


Challenge students to draw a 36-sided polygon with exactly 24 right angles.

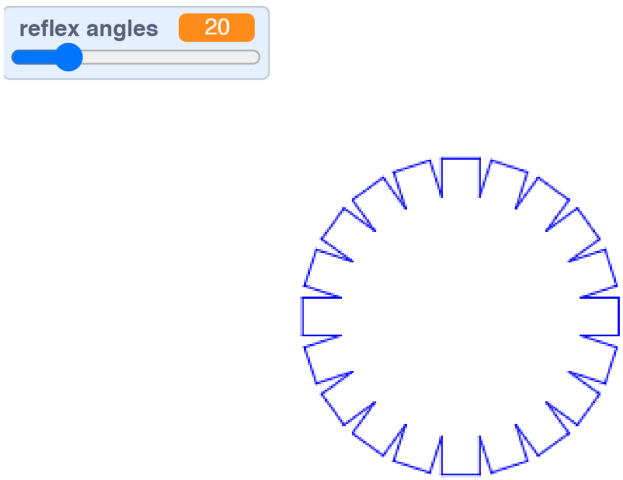
Possible student response:



The code below draws a symmetrical polygon in which two-thirds of the angles are right angles, and one-third are reflex angles.



This code produces the following image for a 60-sided polygon with 20 reflex angles and 40 right angles.



Finding the maximum number of 270° angles

Ask students to find the maximum number of 270° angles for a polygon with seven sides.

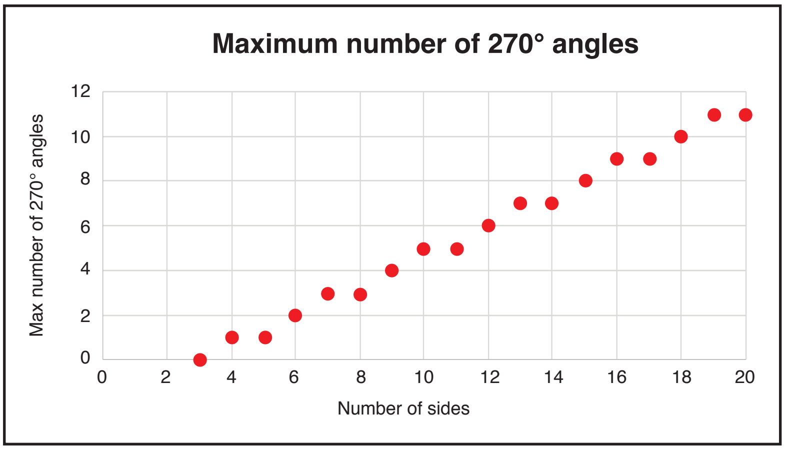
After some trial and error they will be able to make a heptagon with three 270° angles, but not four. Ask them to justify this finding using the formula for the angle sum of a polygon.

Teacher notes:

- The sum of the angles in a heptagon is $180(7 - 2)° = 900°$.
- Four 270° angles would be 1080°, which is more than 900°. Three 270° angles would be 810°, leaving 90° to be shared between the remaining four angles. Therefore, the maximum number of 270° angles is three.

Students can complete the table below and graph their results. Discuss what the graph shows.

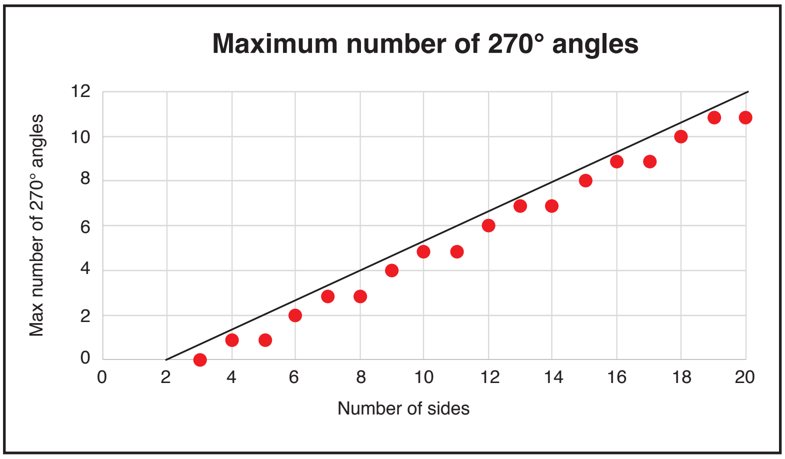
Number of sides	Maximum number of 270° angles
3	0
4	1
5	1
6	2
7	3
8	3
9	4
10	5
11	5
12	6
13	7
14	7
15	8
16	9
17	9
18	10
19	11
20	11



Note that students are solving the inequality $\frac{180(n-2)-270t}{n-t} > 0$, where n is the number of sides and t is the number of 270° angles (i.e. there must be at least one non-270° angle in the polygon).

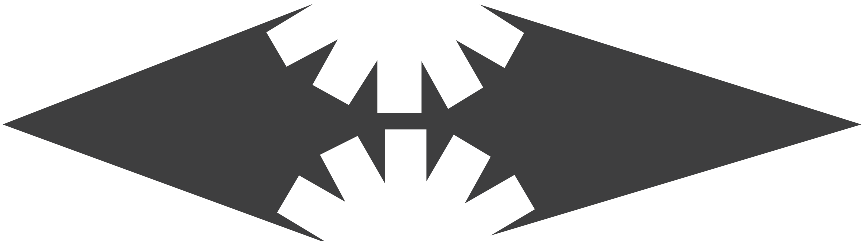
This reduces to $t < \frac{2n-4}{3}$.

The graph on the right includes the line $t = \frac{2n-4}{3}$.



Challenge students to draw a 34-sided polygon with exactly twenty 270° angles.

Possible student response:



It is also possible to draw a 34-sided polygon with twenty-one 270° angles.

Reflection

Discuss the following quote.

> "A mathematician, like a painter or poet, is a maker of patterns. If these patterns are more permanent than theirs, it is because they are made with *ideas*." – G. H. Hardy