re(Solve) MATHS BY INQUIRY Think of a Number

Y9

Summary of learning goals

- Students come to appreciate the power of algebra for generalising results from arithmetic. They connect arithmetic operations with algebraic notation and visualisations. Each lesson commences with an observation made using arithmetic that students then justify and extend using algebra.

Australian Curriculum: Mathematics (Year 9)

ACMNA213: Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate.

Summary of lessons

Who is this sequence for?

- This set of lessons is designed to consolidate skills in algebra, including collecting like terms and expanding and factorising using the distributive law. The resources emphasise the importance of algebra for generalising and justifying arithmetic results. It is assumed that students have some familiarity with algebraic notation.

Lesson 1: Think of a Number (THOAN) – Linear Equations

Lesson 2: Think of a Number (THOAN) – Binomial Equations

- Think of a number (THOAN) activities show how algebraic reasoning can be used to explain and formulate problems in which chains of operations always result in the same outcome regardless of the number chosen. Students are introduced to THOANs involving simple linear operations, and use algebraic simplification to explain why the THOAN works. They are then asked to complete a THOAN when given some starting operations and, finally, to make and test their own.
- In Lesson 1 students work with THOANs using linear equations. Lesson 2 uses binomial equations. The two lessons are otherwise identical.

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

Reflection on this sequence

Rationale

Approaching algebra as generalised arithmetic shows students the power of algebra for abstracting number. This focus on algebra as generalised arithmetic is typically under-represented in secondary mathematics in favour of more time spent on functions and equations.

reSolve mathematics is purposeful

- This sequence supports a rich interpretation and enactment of the Australian Curriculum: Mathematics, providing fun and engaging ways to understand the algebraic content of the Curriculum. The lessons explore mathematics as a creative and imaginative endeavour, emphasising the entertaining applications of mathematics and the understanding of the mathematical foundation beneath common 'tricks' that students are likely to have encountered before.

reSolve tasks are inclusive and challenging

- The tasks in this sequence activate existing knowledge, develop new knowledge and explore relationships between key ideas in the Australian Curriculum. The lessons allow students to determine the complexity of their problems themselves, which allows for a low floor and a high ceiling.

reSolve classrooms have a knowledge-building culture

- Each task in this sequence begins by inspiring curiosity and intrigue through a shared classroom experience that promotes higher-order thinking through the role of both teacher and student. Students build understanding through collaborative inquiry, action and reflection. The sequence encourages students to challenge their existing conceptions and to use their mistakes as a vehicle for further learning.

LESSON 1

re(Solve) MATHS BY INQUIRY

Think of a Number

Think of a Number (THOAN) – Linear Equations

Y9

About this lesson

Think of a number (THOAN) tricks are sequences of operations that result (in this resource) in a return to the starting number regardless of what that number is. In this resource students examine a THOAN and model it visually and algebraically. This develops skills in algebraic operations, including expanding, factorising and collecting like terms. Students see that if the THOAN is to work for any number, it is essential to generate the expressions algebraically. Note that the Create your own THOAN activity can be adapted for any year level and algebraic idea, including polynomials.

Australian Curriculum: Mathematics (Year 9)

ACMNA213: Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate.

Mathematical purpose

- This resource develops students' algebraic skills and provides visual cues to help them develop a deeper understanding of collecting like terms and the distributive law. The context of a THOAN highlights the importance of algebra in showing why a particular set of operations that returns to the starting number will work for any number. The resource gives students confidence in using algebra and a sense of power that they can develop their own number tricks.

Learning intention

- To understand how THOAN tricks work and to create a THOAN trick that works for any given number.

Time

Two to three lessons of approximately 1 hour each.

Vocabulary

- distributive law
- factorise
- like terms

Resources

- reSolve PowerPoint *1a THOAN 1*
- reSolve PowerPoint *1b THOAN 1 with images only*
- reSolve PowerPoint *1c THOAN 1 with algebra only*
- reSolve PDF *THOAN 1 Card Sort*

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

A bit of number 'magic'

Ask students to think of a number (THOAN). Although they may choose any number, it is best to have them choose one that will be easy to work with, such as a small integer. Then, either mentally or with the aid of a calculator, ask them to perform the following operations sequentially:

1. Multiply by 3.
2. Add 5.
3. Subtract the number you first thought of.
4. Subtract 3.
5. Divide by 2.
6. Subtract 1.

The answer should be the number they first thought of.

Teacher notes:

- The algebraic reasoning is shown below, but *do not show the algebra at this stage!*

Instructions	The algebra
1. Think of a number.	x
2. Multiply by 3.	$3x$
3. Add 5.	$3x + 5$
4. Subtract the number you first thought of.	$3x + 5 - x = 2x + 5$
5. Subtract 3.	$2x + 5 - 3 = 2x + 2$
6. Divide by 2.	$\frac{2x+2}{2} = \frac{2(x+1)}{2} = x + 1$
7. Subtract 1.	$x + 1 - 1 = x$

We recommend that you guide the students through two or three more THOAN sequences, but use at least four or five operations each time to show that you are not simply reversing the operations. Although it is more impressive to 'make it up as you go' rather than using prepared examples, two other examples that could be used are:

Instructions	The algebra
1. Think of a number.	x
2. Subtract 5.	$x - 5$
3. Add twice the number you first thought of.	$x - 5 + 2x = 3x - 5$
4. Multiply by 2.	$2(3x - 5) = 6x - 10$
5. Subtract the number you first thought of.	$6x - 10 - x = 5x - 10$
6. Add 10.	$5x - 10 + 10 = 5x$
7. Divide by 5.	$\frac{5x}{5} = x$

Instructions	The algebra
1. Think of a number.	x
2. Multiply by 4.	$4x$
3. Add 11.	$4x + 11$
4. Subtract the number you first thought of.	$4x + 11 - x = 3x + 11$
5. Subtract 2.	$3x + 11 - 2 = 3x + 9$
6. Divide by 3.	$\frac{3x+9}{3} = \frac{3(x+3)}{3} = x + 3$
7. Add twice the number you first thought of.	$x + 3 + 2x = 3x + 3$
8. Divide by 3.	$\frac{3x+3}{3} = \frac{3(x+3)}{3} = x + 1$
9. Subtract 1.	$x + 1 - 1 = x$

Extending prompt:

- *Try starting with a fraction or a negative number.*

Trying your own trick

Ask the class to suggest the first three operations for a THOAN. Collect some suggestions on how they think they can get back to their original number.

For example, consider the following operations:

1. Think of a number.
2. Add 7.
3. Multiply by 4.
4. Subtract 11.

A student who started with 3 will reach 29, and might suggest subtracting 26. A student who started with 5 will reach 37, and might suggest subtracting 7 and dividing by 6. However, these suggestions will not work for everybody. The teacher will have mentally formulated the expression $4x + 17$ and will be able to suggest a set of operations such as:

1. Add the number you first thought of.
2. Subtract 7.
3. Divide by 5.
4. Subtract 2.

Assure the students that by the end of the lesson they will be able to make up their own THOAN magic and astound their friends and family!

Again, do *not* show the algebra at this stage!

Modelling the THOAN

Discuss with students the need to use a variable (rather than a specific number) to find a set of operations that will work for any number.

Resources: The image below is taken from slide 3 of reSolve PowerPoint *1a THOAN 1*.

This shows one way of modelling the THOAN visually, with corresponding algebraic expressions.

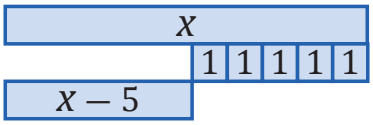
Think of a number.	x	x
Multiply by 3.	x x x	$3x$
Add 5.	x x x 1 1 1 1 1	$3x + 5$
Subtract the number you first thought of.	x x 1 1 1 1 1	$3x + 5 - x = 2x + 5$
Subtract 3.	x x 1 1	$2x + 5 - 3 = 2x + 2$
Factorise.	x 1 x 1	$2x + 2 = 2(x + 1)$
Divide by 2.	x 1	$\frac{2(x+1)}{2} = x + 1$
Subtract 1.	x	$x + 1 - 1 = x$

The PowerPoint guides students through the modelling and algebraic formulation process. Each mouse click or right arrow press on slide 3 allows you to progress from the instruction to the model to the algebraic formulation, and then to the next instruction.

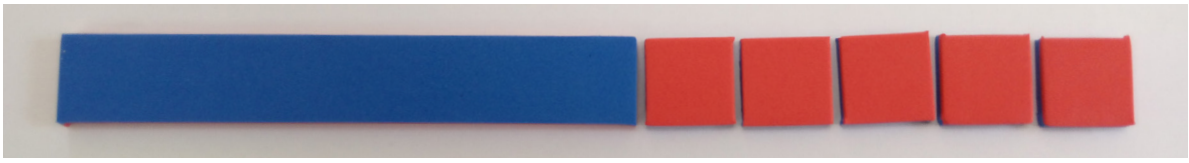
We recommend that the PowerPoint be used as a class demonstration, pausing after each element to ask students what the model will look like and what the algebraic expression will be.

The video reSolve *THOAN 1* (see https://youtu.be/rEdnoQhXHtE) has the same animation with prerecorded timings.

Create two more THOANs that students can model. The two provided provided previously (in the last two tables in the Teacher notes) can be used, but be aware that using subtraction as the first step makes it harder to model. One way of modelling $x - 5$ is shown below.



It is also possible to model the algebra using algebra tiles. The image below shows $x - 5$.



Create your own THOAN

Have students create their own THOAN and test it with the class.

Students can also set up an Excel spreadsheet to test their THOAN.

Enabling prompts:

- For students struggling with the distributive law, use a THOAN such as:
  1. Think of a number.
  2. Multiply by 2.
  3. Add 6.
  4. Divide by 2.
- *What number do you need to add to get back to the number you first thought of? Why?*

Extending prompt:

- Encourage students to use more operations. (Although they will hardly need any encouragement!)

Discussion

Discuss with students the limitations of the modelling method. These might include the difficulty of dealing with negative numbers or with fractions. They may also include the limitation of the modelling to one dimension, such as for linear functions (although the reSolve PowerPoint *2a THOAN 2* moves into two dimensions, involving binomial expansions and the use of area diagrams).

Discuss the use of the distributive law and why it is important to factorise before dividing. Show how the use of the visual models helps to illustrate expansion and factorisation using the distributive law. Make connections to the reSolve Year 8 resource Tens and Units, in which the connection is made between the distributive law and multiplication algorithms.

Further activities

Activity 1: Complete the THOAN

Give students some instructions that are the start of a THOAN; for example:

1. Think of a number.
2. Subtract 3.
3. Multiply by 5.
4. Add 5.

Ask students to complete the THOAN in two steps. *Is there more than one way to do this? What is the connection between these two different methods?*

The result so far is $5x - 10$. So two possible ways to complete the THOAN are to add 10 and divide by 5, or divide by 5 and add 2. Note that there are other possibilities, such as subtracting 4, then multiplying by the original number and adding 10.

Ask students to work in pairs and try this for themselves, with one person giving three or four initial steps and the other completing the THOAN.

Ask students to complete a THOAN in three steps and compare the different answers.

Activity 2: A take-home THOAN

Suggest that students try this at home and explain to their family how it works.

Activity 3: A different end point

Ask students to create a THOAN for which the result is 5, no matter what number they started with.

Activity 4: Start with the diagrams

Give students a set of diagrams that give a visual representation of a THOAN and ask them to complete the list of instructions and the algebraic formulation.

Resources: The reSolve PowerPoint *1b THOAN 1 with images only* is an example.

Activity 5: Start with the algebra

Give students the algebraic formulations and ask them to complete the list of instructions and the visual representations.

Resources: The reSolve PowerPoint *1c THOAN 1 with algebra only* is an example.

Activity 6: Card sort

Use a set of cards that will result in a THOAN; mix them up and ask students to rearrange them to make it work.

Resources: The reSolve PDF *THOAN 1 Card Sort* is an example.

LESSON 2

re(Solve) MATHS BY INQUIRY Think of a Number

Think of a Number (THOAN) – Binomial Equations

Y9

About this lesson

Think of a number (THOAN) tricks are sequences of operations that result (in this resource) in a return to the starting number regardless of what that number is. In this resource students examine a THOAN and model it visually and algebraically. This develops skills in algebraic operations, including expanding, factorising and collecting like terms. Students see that if the THOAN is to work for any number, it is essential to generate the expressions algebraically. Note that the Create your own THOAN activity can be adapted for any year level and algebraic idea, including polynomials.

Australian Curriculum: Mathematics (Year 9)

ACMNA213: Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate.

Mathematical purpose

- This resource develops students' algebraic skills and provides visual cues to help them develop a deeper understanding of collecting like terms and the distributive law. The context of a THOAN highlights the importance of algebra in showing why a particular set of operations that returns to the starting number will work for any number. The resource gives students confidence in using algebra and a sense of power that they can develop their own number tricks.

Learning intention

- To understand how THOAN tricks work and to create a THOAN trick that works for any given number.

Time

Two to three lessons of approximately 1 hour each.

Vocabulary

- distributive law
- factorise
- like terms

Resources

- reSolve PowerPoint *2a THOAN 2*
- reSolve PowerPoint *2b THOAN 2 with images only*
- reSolve PowerPoint *2c THOAN 2 with algebra only*
- reSolve PDF *THOAN 2 Card Sort*

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

A bit of number 'magic'

Ask students to think of a number (THOAN). Although they may choose any number, it is best to have them choose one that will be easy to work with, such as a small integer. Then, either mentally or with the aid of a calculator, ask them to perform the following operations sequentially:

1. Add 3 and remember your answer.
2. Add 2 to your original number and remember your answer.
3. Multiply these two answers together.
4. Subtract 6.
5. Divide by your original number.
6. Subtract 5.

The answer should be the number they first thought of.

Teacher notes:

- The algebraic reasoning is shown below but *do not show the algebra at this stage!*

Instructions	The algebra
1. Think of a number.	x
2. Add 3 and remember your answer.	$x + 3$
3. Add 2 to your original number and remember your answer.	$x + 2$
4. Multiply these two answers together.	$(x + 3)(x + 2) = x^2 + 5x + 6$
5. Subtract 6.	$x^2 + 5x + 6 - 6 = x^2 + 5x$
6. Divide by your original number.*	$\frac{x^2 + 5x}{x} = \frac{x(x + 5)}{x} = x + 5$
7. Subtract 5.	$x + 5 - 5 = x$

* If the original number was zero, then this step will give an answer that is undefined.
You might like to start by saying, "Think of a non-zero number."
We recommend that you repeat this process two or three times, but use at least four or five operations each time to show that you are not simply reversing the operations. Although it is more impressive to 'make it up as you go' rather than using prepared examples, two other examples that could be used are:

Instructions	The algebra
1. Think of a number.	x
2. Add 3.	$x + 3$
3. Square your answer.	$(x + 3)^2 = x^2 + 6x + 9$
4. Add the number you first thought of.	$x^2 + 6x + 9 + x = x^2 + 7x + 9$
5. Subtract 3.	$x^2 + 7x + 9 - 3 = x^2 + 7x + 6$
6. Divide by 1 more than the number you first thought of.*	$\frac{x^2 + 7x + 6}{x + 1} = \frac{(x + 6)(x + 1)}{x + 1} = x + 6$
7. Subtract 6.	$x + 6 - 6 = x$

* Note that if a student's original number was −1, the result at this step will be undefined.

Instructions	The algebra
1. Think of a number.	x
2. Add 3.	$x + 3$
3. Multiply by the number you first thought of.	$x\,(x + 3) = x^2 + 3x$
4. Subtract 10.	$x^2 + 3x - 10$
5. Divide by 5 more than the number you first thought of.*	$\frac{x^2 + 3x - 10}{x + 5} = \frac{(x + 5)(x - 2)}{x + 5} = x - 2$
6. Multiply by 1 more than the number you first thought of.*	$(x - 2)(x + 1) = x^2 - x - 2$
7. Add the number you first thought of.	$x^2 - x - 2 + x = x^2 - 2$
8. Add 2.	$x^2 + 2 = x^2$
9. Find the square root.§	$\sqrt{x^2} = \lvert x \rvert$

* If a student's starting number was −5, the result at this step will be undefined.

§ If a student's starting number was negative, finding the square root at this step will give a positive answer. Hence, technically we write this as $|x|$.

Trying your own trick

Ask the class to suggest the first three or four operations, at least one of which involves squaring the number or multiplying by an expression that involves the original number, and collect some suggestions on how they think they can get back to their original number.

For example, if the four operations are:

1. Think of a number.
2. Subtract 4 and remember your answer.
3. Add 1 to your original number and remember your answer.
4. Multiply these answers together.
5. Add 6.

Then a student who started with 3 will reach 2, and might suggest subtracting 2. A student who started with 10 will reach 72, and might suggest subtracting 2, then dividing by their original number and adding 3. However, these suggestions will not work for everybody.

The teacher will have mentally formulated the expression $x^2 - 3x + 2$ or $(x - 2)(x - 1)$ and will be able to suggest a set of operations such as:

6. Divide by 1 less than your original number.
7. Add 2.

Assure the students that by the end of the lesson they will be able to make up their own THOAN magic and astound their friends and family!

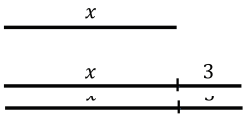
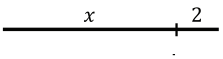
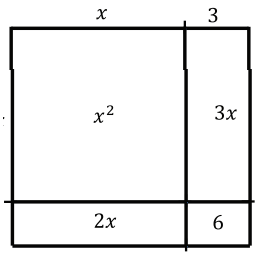
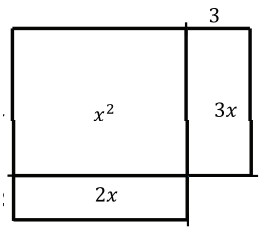
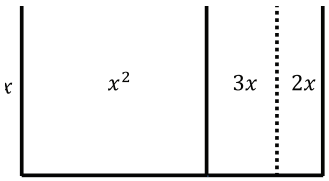
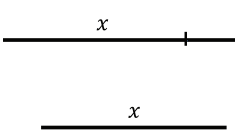
Again, do *not* show the algebra at this stage!

Modelling the THOAN

Discuss with students the need to use a variable (rather than a specific number) to find a set of operations that will work for any number.

Resources: The image below is taken from slides 3 and 4 of reSolve PowerPoint *2a THOAN 2*.

This shows one way of modelling the THOAN visually, with corresponding algebraic expressions.

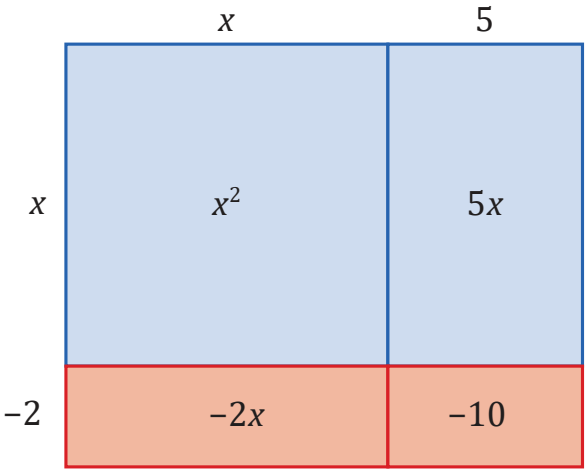
Think of a number.		x
Add 3 and remember your answer.		$x + 3$
Add 2 and remember your answer.		$x + 2$
Multiply these two answers together.		$(x + 3)(x + 2) = x^2 + 5x + 6$
Subtract 6.		$x^2 + 5x + 6 - 6 = x^2 + 5x$
Factorise.		$x^2 + 5x = x(x + 5)$
Divide by the number you first thought of.		$\frac{x(x+5)}{x} = x + 5$
Subtract 5.		$x + 5 - 5 = x$

The PowerPoint guides students through the modelling and algebraic formulation process. Each mouse click or right arrow press on slides 3 and 4 allows you to progress from the instruction to the model to the algebraic formulation, and then to the next instruction. The image at the end of slide 3 is repeated at the start of slide 4 for ease of reference.

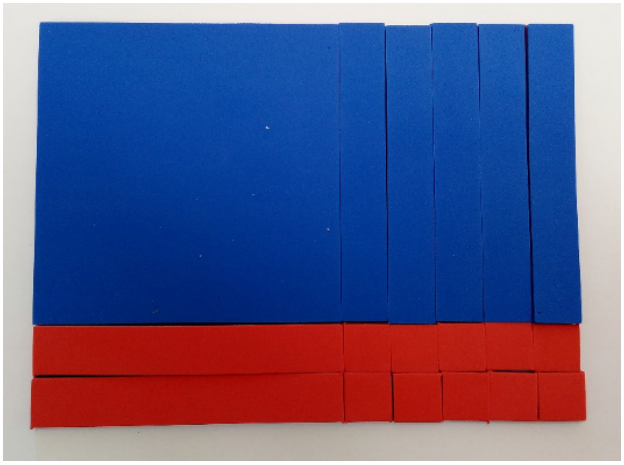
We recommend that the PowerPoint be used as a class demonstration, pausing after each element to ask students what the model will look like and what the algebraic expression will be.

The video reSolve *THOAN 2* (see https://youtu.be/oHbz3cV0EUg) has the same animation with prerecorded timings.

Create two more THOANs that students can model. In each case, ask students to find a number for which the THOAN will not work. The two provided previously (in the last two tables in the Teacher notes) can be used; however, the subtraction in the last example makes it harder to model. One way to model $x^2 + 3x - 10$ is shown below. Note that this anticipates the factorisation by rewriting the binomial as $x^2 + 5x - 2x - 10$.



This can also be modelled using algebra tiles.



Create your own THOAN

Have students create their own THOAN and test it with the class.

Students can also set up an Excel spreadsheet to test their THOAN.

Enabling prompts:

- For students struggling with the distributive law, use a THOAN such as:
  1. Think of a number.
  2. Add 2 to the number and remember your answer.
  3. Add 3 to the number and remember your answer.

  Now ask:
  - ◊ What number do you need to subtract to remove the constant term?
  - ◊ What term do you need to subtract to leave only x^2?
  - ◊ What can you do now to get back to your original number?

Extending prompt:

- Encourage students to use more operations. (Although they will hardly need any encouragement!)

Discussion

Discuss with students the limitations of the modelling method. These might include the difficulty of dealing with negative numbers or with not being able to show invalid results.

Discuss the use of the distributive law and why it is important to factorise before dividing. Show how the use of the visual models helps to illustrate expansion and factorisation using the distributive law. Make connections to the reSolve Year 8 resource Tens and Units, in which the connection is made between the distributive law and multiplication algorithms.

Emphasise to students that division by zero will give an undefined result, and that finding the square root will give only positive answers. This means that if the number chosen is such that it results in division by zero, the THOAN will not work; and if the required result from finding a square root is negative, the THOAN will not work. We can include this in the algebraic formulation, as given below.

$$\frac{x^2+3x-10}{x+5} = \frac{(x+5)(x-2)}{x+5} = x-2, x \neq -5$$

Further activities

Activity 1: Complete the THOAN

Give students some instructions that are the start of a THOAN; for example:

1. Think of a number.
2. Square it.
3. Subtract twice your original number.
4. Subtract 15.

Ask students to complete the THOAN in two steps. *Is there more than one way to do this? For what number(s) will your THOAN give an undefined result?*

The result so far is $x^2 - 2x - 15 = (x-5)(x+3)$. So two possible ways to complete the THOAN are to divide by 5 less than your original number and then subtract 3, or to divide by 3 more than the original number and then add 5. In the first case, the THOAN will not work if the starting number is 5, and in the second case it will not work if the starting number is -3.

Ask students to work in pairs and try this for themselves, with one person giving three or four initial steps and the other completing the THOAN.

Ask students to complete a THOAN in three steps and compare the different answers.

Activity 2: A take-home THOAN

Suggest that students try this at home and explain to their family how it works.

Activity 3: A different end point

Ask students to create a THOAN for which the result is 5, no matter what number they started with.

Activity 4: Start with the diagrams

Give students a set of diagrams that give a visual representation of a THOAN and ask them to complete the list of instructions and the algebraic formulation.

Resources: The reSolve PowerPoint *2b THOAN 2* with images only is an example.

Activity 5: Start with the algebra

Give students the algebraic formulations and ask them to complete the list of instructions and the visual representations.

Resources: The reSolve PowerPoint *2c THOAN 2* with algebra only is an example.

Activity 6: Card sort

Use a set of cards that will result in a THOAN; mix them up and ask students to rearrange them to make it work

Resources: The reSolve PDF *THOAN 2 Card Sort* is an example.