re(Solve) MATHS BY INQUIRY Cartesian Product

Y4

Summary of learning goals

- The sequence introduces the key idea of multiplication as a Cartesian product, using the language of 'for each'. In the first task, students learn to use a tree diagram to find the number of possible combinations that can be made in an animal mix-and-match book.
- Students learn how a simpler problem can be used to help solve a larger, more complex problem. In the second task, students design an avatar that has two or three different features and work as a class to find the total number of possible avatars. They look at how all combinations can be represented as an array and then they learn how to create a tree diagram.

Australian Curriculum: Mathematics (Year 4)

ACMNA075: Recall multiplication facts up to 10×10 and related division facts.

ACMNA076: Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder.

Summary of lessons

Who is this sequence for?

- It is assumed that students have previously used arrays to help solve multiplication problems and that they are building their recall of multiplication facts up to 10×10. It would also be helpful if students have worked on multiplication problems that involve multiplying three or four numbers together and questions that use at least one multidigit number.

Lesson 1: What is a Plocoroo?

Students explore the number of animals it is possible to make using a mix-and-match book. The book contains 10 different animals that have each been divided into three parts: head, body and tail. To help solve the larger problem, students are encouraged to use a simpler problem. They use a tree diagram in their working.

Lesson 2: Making Avatars

Students design their own avatar by choosing options from a set of features. The class is asked to consider if all possible avatars have been made, given the different features that can be selected. Students use an array and are then introduced to a tree diagram to explore the Cartesian product for multiplication.

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

Reflection on this sequence

Rationale

Multiplication is a central element of number in the primary years of school. Much time is spent building the concepts of rate and equal groups, particularly through the array representation. Another important concept, yet often overlooked, is that of 'for each'. This sequence uses Cartesian product to explore this idea. The sequence focuses on developing this idea through representation rather than symbolisation. A tree diagram is used as a means to model all possible combinations and, through this diagram, students learn that the number of features in each category are multiplied to find the total number of combinations that can be made.

reSolve mathematics is purposeful

- The sequence focuses on the important multiplicative concept of 'for each'.
- Uses playful contexts for Cartesian product that can be easily imagined by students.

reSolve tasks are inclusive and challenging

- Students learn that a large problem can be made more accessible by using a smaller, simpler problem.
- Students can vary the magnitude of numbers they are working with to challenge themselves appropriately.

reSolve classrooms have a knowledge-building culture

- Students create their own avatars and animals to compile a class collection.
- Students work collaboratively to model the problem.

LESSON 1

re(Solve) MATHS BY INQUIRY Cartesian Product

What is a Plocoroo?

Y4

About this lesson

Students explore the number of animals it is possible to make using a mix-and-match book. The book contains 10 different animals that have each been divided into three parts: head, body and tail. To help solve the larger problem, students are encouraged to use a simpler problem. They use a tree diagram in their working.

Australian Curriculum: Mathematics (Year 4)

ACMNA075: Recall multiplication facts up to 10 × 10 and related division facts.

ACMNA076: Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder.

Mathematical purpose

- The resource explores the key idea of multiplication as a Cartesian product, using the language of 'for each'. Students learn to use a tree diagram and how a simpler problem can be used to help solve a larger, more complex problem.

Learning intention

- To find how many different animals can be made using a mix-and-match book.

Time

A lesson of approximately 1 hour.

Vocabulary

- Cartesian product
- for each

Resources

- reSolve PowerPoint *1a What is a Plocoroo?*
- reSolve PDF *1b What is a Plocoroo?* printed, bound and cut into three parts so that different animals can be made. If possible, create multiple copies so that the students can explore and experiment.
- reSolve PDF *1c Animal Names* (optional: see Teacher notes in Reflection for preparation)

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Making animals

Resources: Show the book *1b What is a Plocoroo?* to the students and look at some of the different animals that can be made.

Explain that the book has been made using 10 different animals, with each animal cut into three parts.

Pose the question: *How many different animals can be made using this book?*

Allow the students to start exploring the problem.

Using a simpler problem

After some time has been allowed for student experimentation, discuss the complexity of working with all 10 animals. This might be a class discussion or conversations with students as the teacher moves around the classroom. Explain that mathematicians will often solve a simpler problem and use what they learn to help them solve a larger problem.

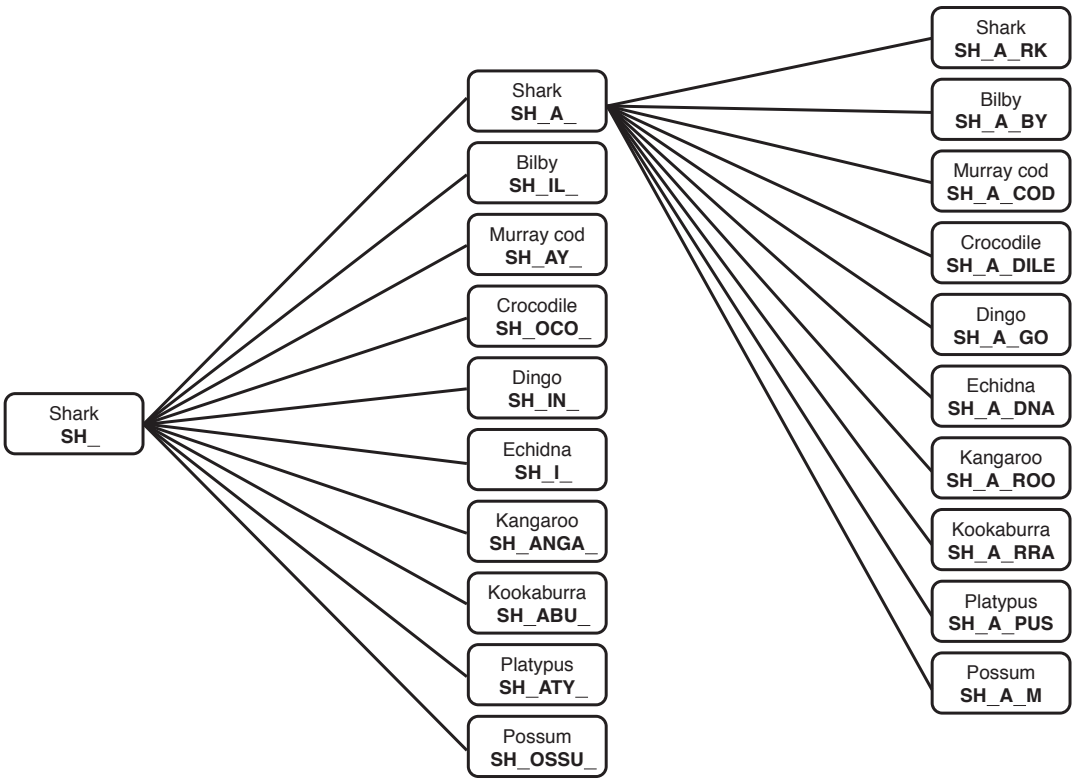
Encourage the students to think about how a simpler problem might be used to help solve the problem.

Two approaches that may simplify the problem:

- Start with one head and consider possible combinations.
- What if there were just two animals? What if there were three?

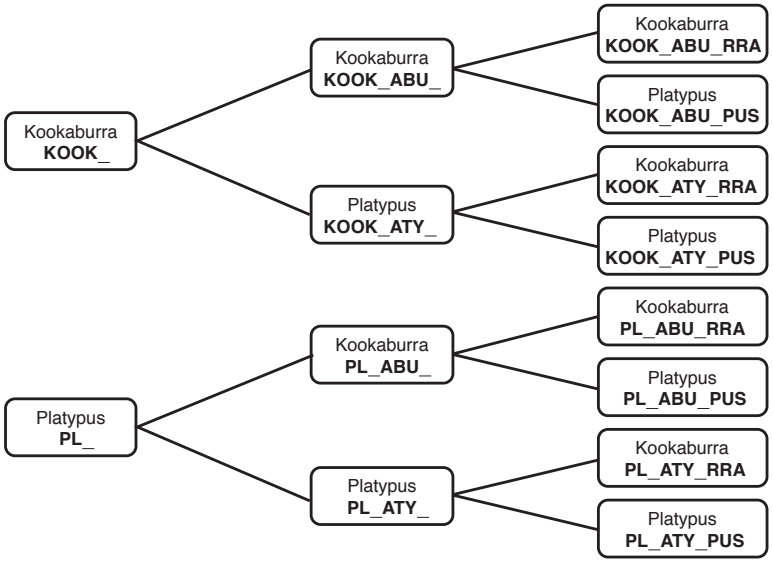
Possible student responses:

- *Let's look at all the possible combinations for one head.*
  - ◊ Starting with the shark head, there are **10** possible bodies. **For each** body there are **10** tails.
  - ◊ Generalising, **for each** head in the book there are 10 possible bodies: 10 heads × 10 bodies. **For each** of these bodies there are 10 tails: 10 heads × 10 bodies × 10 tails.

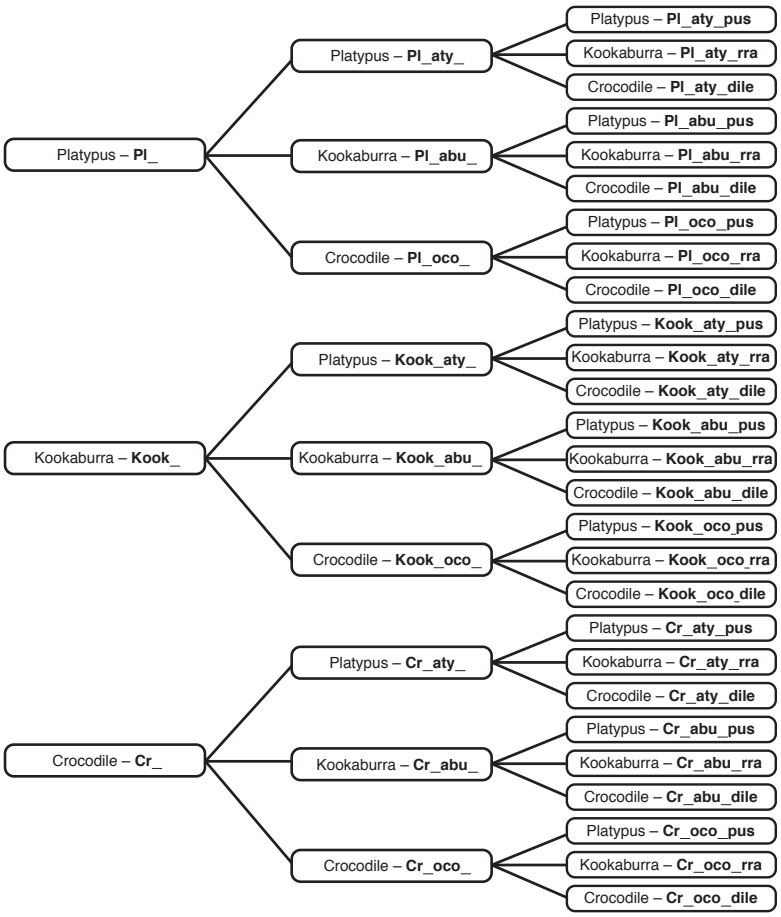


- *What if there is just one animal?*
  - ◊ **For each** head there is only one body: 1 head × 1 body.
    For each body there is only one tail: 1 head × 1 body × 1 tail. This gives one animal.

- *What if there are two animals?*
  - ◊ **For each** head there are two bodies: 2 heads × 2 bodies. **For each** of these bodies there are two tails: 2 heads × 2 bodies × 2 tails. This gives eight different animals.



- *What if there are three animals?*
  - ◊ **For each** head there are three bodies: 3 heads × 3 bodies. **For each** of these bodies there are three tails: 3 heads × 3 bodies × 3 tails. This gives 27 different animals.



- *What if there are 10 animals?*
  - **For each** head there are three bodies: 10 heads × 10 bodies. **For each** of these bodies there are ten tails: 10 heads × 10 bodies × 10 tails. This gives 1000 different animals.

Reflection

Select students to share their work with the class.

T Teacher notes:

- It is possible to make a human tree diagram using the cards in the reSolve PDF *1c Animal Names*. The resource provides name cards for three animals: the kookaburra, platypus and crocodile.
- If you would like to make all the possible combinations for just two animals (e.g. the kookaburra and platypus), make:
  - one copy of the kookaburra head and one of the platypus head (pages 1 and 2)
  - two copies of each body (pages 4 and 5)
  - four copies of each tail (pages 7 and 8).
- To make all combinations for three animals, you will need:
  - one copy of each head (pages 1–3)
  - three copies of each body (pages 4–6)
  - nine copies of each tail (pages 7–9).
- It would require 39 people to create a human tree diagram for all combinations with three animals (3 people for the heads + 9 for the bodies + 27 for the tails). The cards could be placed on the floor instead to show all the combinations.
- Give the head cards to two students.
  - Ask: *How many body cards will be needed to show all the different combinations of heads and bodies?* Hand out the body cards and ask students to show all the combinations.
  - Ask: *How many tail cards will be needed to show all the different combinations of heads, bodies and tails?* Hand out the tail cards and ask students to show all the combinations.

Ask students:

- *How many combinations would be possible with 2 heads, 3 bodies and 4 tails?*
- *How many combinations would be possible with 10 heads, 9 bodies and 8 tails?*
- *How many heads, bodies and tails would be needed to make 60 to 70 different combinations? What about 100 to 150?*

Where to next?

In Lesson 2: Making Avatars students design their own avatar by choosing options from a set of features, then consider if all possible combinations of the features have been made.

LESSON 2

re(Solve) MATHS BY INQUIRY Cartesian Product

Making Avatars

Y4

About this lesson

Students design their own avatar by choosing options from a set of features. The class is asked to consider if all possible avatars have been made, given the different features that can be selected. Students use an array and are then introduced to a tree diagram to explore the Cartesian product for multiplication.

Australian Curriculum: Mathematics (Year 4)

ACMNA075: Recall multiplication facts up to 10 × 10 and related division facts.

ACMNA076: Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder.

Mathematical purpose

- To introduce the key idea of multiplication as a Cartesian product, using the language of 'for each'.

Learning intention

- To find how many different avatars can be made by changing only two or three features.

Time

A lesson of approximately 1 hour.

Vocabulary

- avatar
- Cartesian product
- for each

Resources

- *Guess Who?* board game (optional)
- reSolve PDF *2a Avatar Templates* (one avatar per student)
- string

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Make your avatar

Resources: If you have a game of *Guess Who?*, play it.

In the game of *Guess Who?*, not every possible combination of facial features is used. If all the possible combinations were included, how many different people might there be?

Explain to the students that you will be making a classroom version of *Guess Who?*, using avatars that they will create themselves.

Resources: Have each student design and name an avatar using the template provided in reSolve PDF *2a Avatar Templates*.

Look at the students' avatars and create a list of categories used. For example:

- Hair colour – blonde, brown, red, black
- Eye colour – blue, green, hazel, brown
- Hair shape – long, short, curly, bald
- Mouth – smile, frowning, tongue sticking out
- Accessories – glasses, hat, beard.

As a class, choose **two** categories (e.g. hair colour and eye colour).

Pose the questions: *Has the class drawn every possible avatar using these two categories? If we have them all, how do we know? If there are some missing, what are they?*

Finding all avatars

Ask students to hold onto their avatars and to arrange themselves so that they are standing next to someone who has the same feature in one of the chosen categories (e.g. standing next to someone with the same hair colour or eye colour). Encourage students to think about different ways to arrange themselves in columns and in rows.

Eventually this will result in a 2-dimensional array, with each class member standing in an appropriate position. Some positions may have more than one student if they have designed an avatar with the same features. If some positions are vacant, ask a student who is doubled up to make an avatar that would fit in that position.

	Hair colour			
Eye colour	**Blonde**	**Brown**	**Red**	**Black**
Brown	 Emily	 Katie	 Tom	 Georgie
Blue	 Sophie	 Henry	 Chris	 Mia
Green	 William	 Joe	 Charlotte	 Jack

Students can record their results by using their avatars as an array on the board.

Explain to students that **for each** of the four different hair colours there are three different eye colours: $4 \times 3 = 12$ avatars. Alternatively, **for each** of the three different eye colours there are four different hair colours: $3 \times 4 = 12$ avatars.

Ask students related questions such as: *How many different avatars would there be if there were five possible hair colours and three possible eye colours? How many possible eye colours and hair colours might there be if the total number of different avatars was 18?*

Accessorising avatars

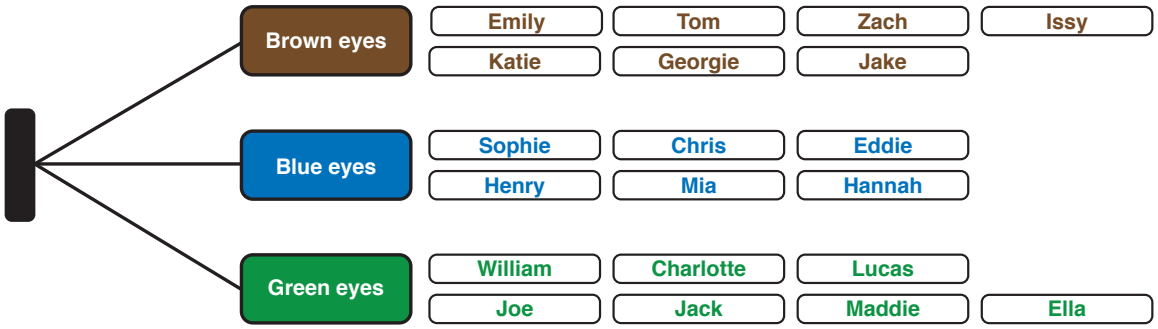
Tell the students that we are now going to add another element, such as accessories. Ask students to predict the number of different avatars possible if there are three possible eye colours, four possible hair colours and two possible accessories (e.g. glasses or hat). For the purpose of this task an avatar *must* have an accessory.

Possible student response:

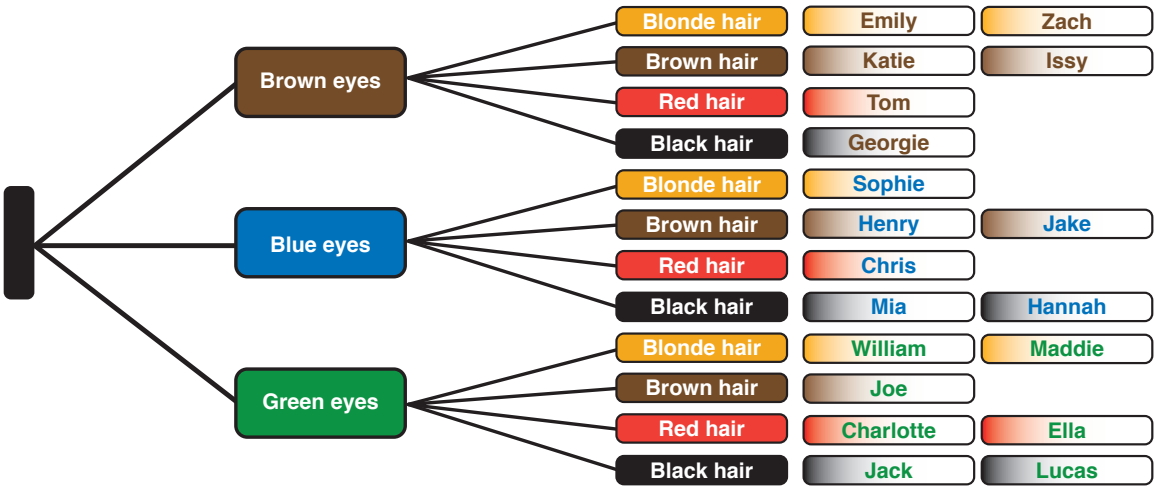
- **For each** of the avatars we have already made there are two different mouths they can have, so there are $3 \times 4 \times 2 = 24$ different avatars.

Ask the students to work together to make all of the possible avatars and see if their prediction is correct.

The challenge now is to represent the avatars so that we are sure that we have them all. One way is to make a human tree diagram. Ask students to make groups with the same eye colour. Use string and some labels to show the three groups (e.g. as shown below for 20 students). Ask students how the tree diagram helps to show which avatars have which eye colour.



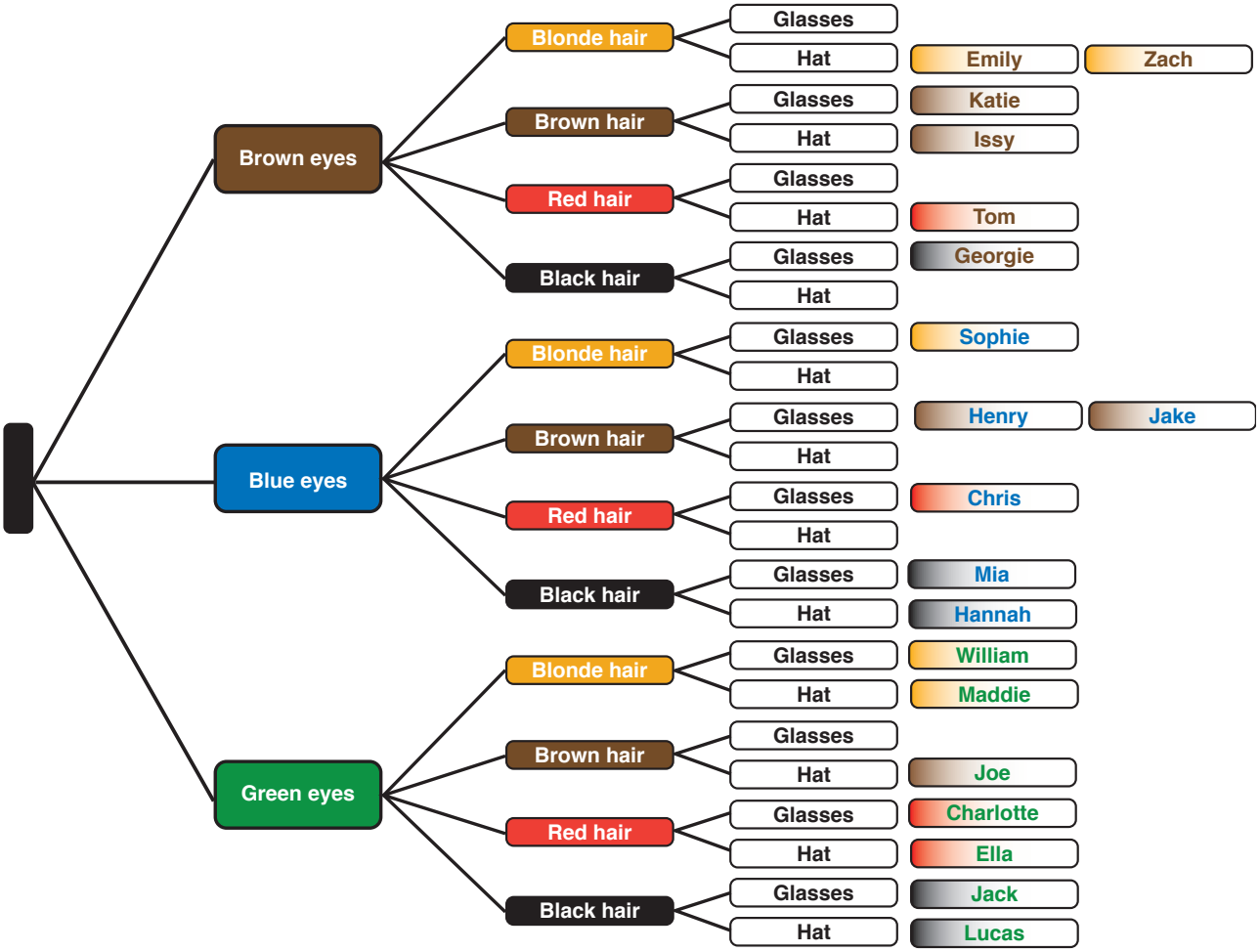
Now within each group ask the students to make groups with the same hair colour. For example:



Make sure that students understand how the tree diagram shows the possible combinations.
For example, ask how many avatars have both blonde hair and brown eyes, or what features a particular student's avatar has.

Accessorising avatars

Now add a third layer to show accessories, as demonstrated below.



Ask students questions such as: *How many avatars have red hair, brown eyes and a hat? What facial features does [student's name] have? Which combinations have more than one avatar? Which combinations have no avatars?*

Reflection

Select some students to share their work with the class.

Ask students how many possible avatars there would be if we had one of every possible combination with no duplicates.

Ensure that students understand that **for each** of the eye colours there are four hair colours, and **for each** combination of eye and hair colour there are two features: $3 \times 4 \times 2 = 24$ possible avatars.

Ask students how else they might be able to draw the diagram. For example, we could start with hair colour, then eye colour, then features, and get $4 \times 3 \times 2 = 24$ possible avatars.

A game of *Guess Who?* could be played using the class's avatars.