Year 8: Quadrilateral Linkages
Lesson 3: Folding Umbrella

Australian Curriculum: Mathematics (Year 8)

ACMMG201: Develop the conditions for congruence of triangles.
- Solving problems using the properties of congruent figures.

ACMMG202: Establish properties of quadrilaterals using congruent triangles and angle properties.
- Establishing the properties of squares, rectangles, parallelograms, rhombuses, trapeziums and kites.
- Identifying properties related to side lengths, parallel sides, angles, diagonals and symmetry.

Lesson abstract

Students investigate the design and operation of a folding umbrella to see how the geometric properties of a parallelogram contribute to neat folding. Students observe an actual umbrella and its ribs, make a physical model and operate a computer simulation, at each stage seeing the geometric features more fully. This linkage is then used as a stimulus for students to engage in deductive reasoning about the properties of quadrilaterals.

Mathematical purpose (for students)

If a quadrilateral is designed with opposite sides equal, then it will be a parallelogram.

Mathematical purpose (for teachers)

This folding umbrella provides an engaging context for students to analyse a real object geometrically, and to engage in deductive reasoning. Hands-on experience with an umbrella and a physical model supports use of geometric language and making conjectures. Using the computer simulation enables these conjectures to be tested. Students then reason deductively, with visual support by a slide show of diagrams, about the key geometric property for neat folding. A parallelogram is defined as a quadrilateral with opposite sides parallel, and it follows that the opposite sides are also equal. Here, we start with the converse –the umbrella ribs include a quadrilateral made so that the opposite sides are equal. It can be shown that the opposite sides are therefore parallel.

Lesson Length 50-60 minutes approx. or 2 hours if combined with Toolbox and Cherry picker lessons

Vocabulary Encountered
- congruent
- diagonal
- converse

Lesson Materials
- Metal ribs from an old umbrella
- Geostrips or similar (2 each of two different lengths per pair of students, or print these strips onto card).
- Paper fasteners (4 per pair of students)
- Student Sheet 1 – Folding Umbrella (1 per student)
- Slide show: *ST1_Yr8_3a_UmbrellaImages.pptx*
- Slide show: *ST1_Yr8_3c_IsItParallelogram.ppt* (all students to access)
- GeoGebra file *ST1_Yr8_3b_Umbrella.ggb*
- OR GeoGebra Tube link https://ggbm.at/fAHGQpqp

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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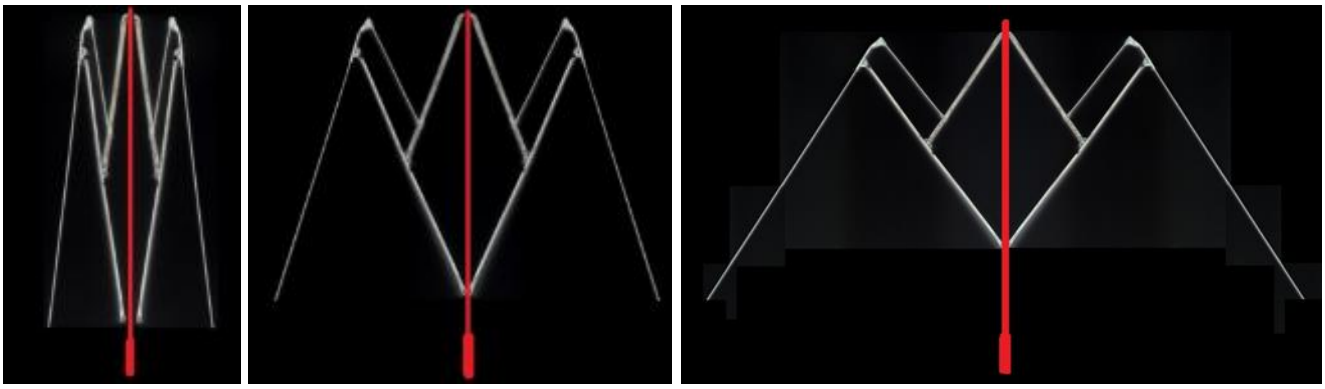
Examining a Folding Umbrella

This lesson is based on a folding umbrella, sometimes called a pocket or telescopic umbrella.

The metal frames of most folding umbrellas have eight ribs that radiate out from the centre. Each rib is made up of hinged bars that allow the umbrella to fold.

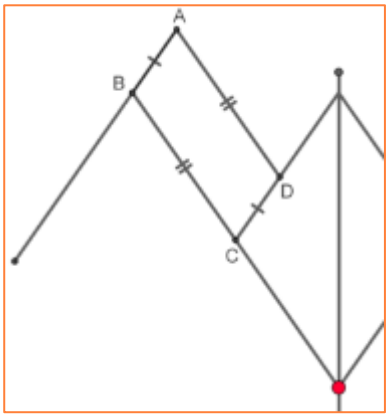
If you have an old umbrella, it can be pulled apart to separate the eight ribs (three old umbrellas gives a class set!).

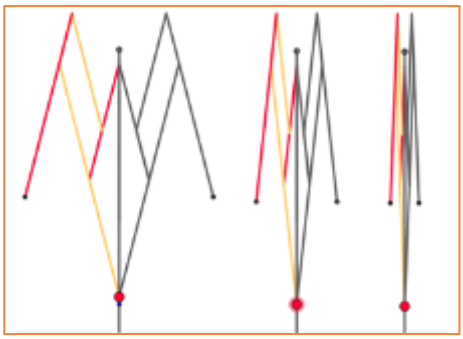
The images below show how the hinged bars of one of the ribs of a two-fold umbrella move as the umbrella is opened. This umbrella folds to half its length.



Teacher Notes

- The key to folding is having the parallelograms along the 'broken' ribs. (See parallelogram ABDC in the diagram.) These keep the rib sections parallel to each other.
- Each parallelogram is constructed from rib sections of fixed length, making both pairs of opposite sides of the quadrilateral equal in length.
- The definition of a parallelogram is 'a quadrilateral with both pairs of opposite sides parallel'. It can then be proved that both pairs of opposite sides are equal.
- The converse is also true – if a quadrilateral has both pairs of opposite sides **equal**, it is always a **parallelogram** (see Teacher Sheet 1 – Parallelograms and Deductive Reasoning.
- As the umbrella is opened and closed, the pairs of rods, *AB* and *DC*, and *AD* and *BC* remain parallel.
- When the umbrella is nearly folded, the central parts of the ribs are nearly parallel to the central pole. So all parts of the ribs are also nearly parallel to the central pole. This makes neat folding.





First observations of folding umbrella ribs

- If possible, provide each group of students with an actual umbrella rib obtained by pulling apart an old umbrella (eight ribs from each umbrella). Show students the first 3 slides from slide show *ST1_Yr8_3a_UmbrellaImages.pptx* so that they can see the shapes clearly.
- Discuss what the umbrella ribs are for and how they work. What are their key features?

Expected Student Responses

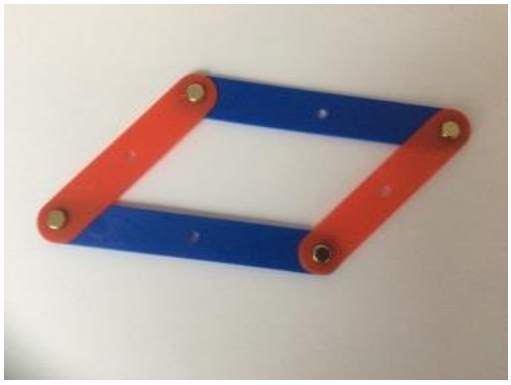
- The ribs are part of the frame that holds the umbrella up.
- The ribs are hinged so that umbrella can fold.
- Each rib includes a parallelogram that helps the umbrella fold neatly.

Is the quadrilateral really a parallelogram?

Ask students how the parallelograms have been made by the linkages. (ANS: They are parallelograms because the side lengths are fixed so that both pairs of opposite sides have equal length.

Give each pair of students 4 strips, 2 each of two different lengths, and 4 paper fasteners. This can be done with Geo Strips or with card strips.

Constructing a quadrilateral with both pairs of opposite sides equal allows students to see that the opposite sides stay parallel as they change the shape of the quadrilateral. This is proved later in the lesson.

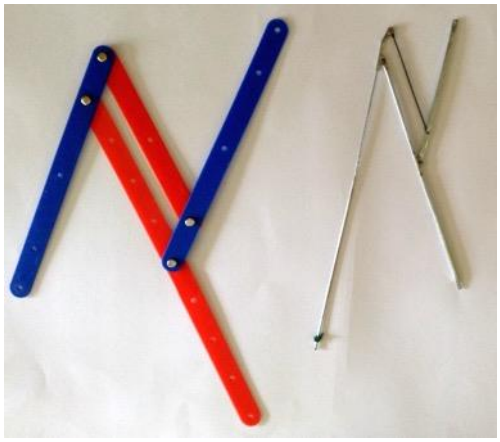


Models of the Folding Umbrella

Constructing a physical model

Students can use Geo Strips to produce a model of the umbrella linkage, making sure they choose Geo Strips that give equal pairs of opposite sides for the quadrilateral. Alternatively, these strips can be cut from card.

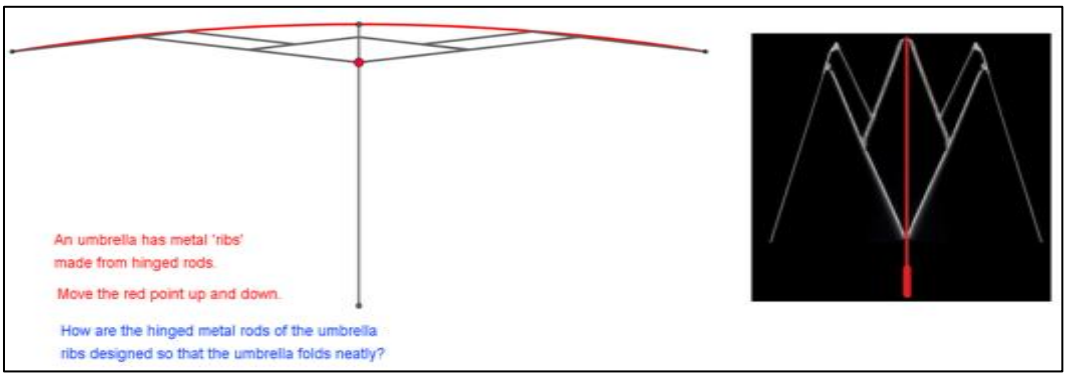
The main point here is that making the opposite sides of the parallelogram of equal length ensures that the opposite sides are parallel.

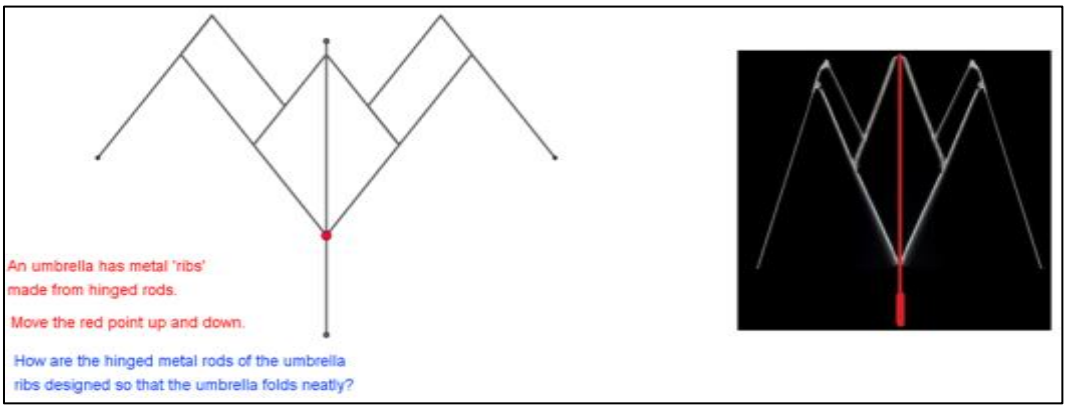


Using a computer simulation

The GeoGebra file file *ST1_Yr8_3b_Umbrella.ggb* (or the GeoGebra Tube link https://ggbm.at/fAHGQpqp) provides an interactive simulation of the opening and closing of a folding umbrella.

Students can check conjectures using this file (visually or by measuring). The main question is to explore what geometric features make the umbrella fold neatly.



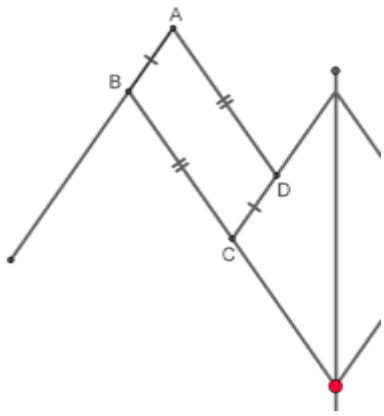


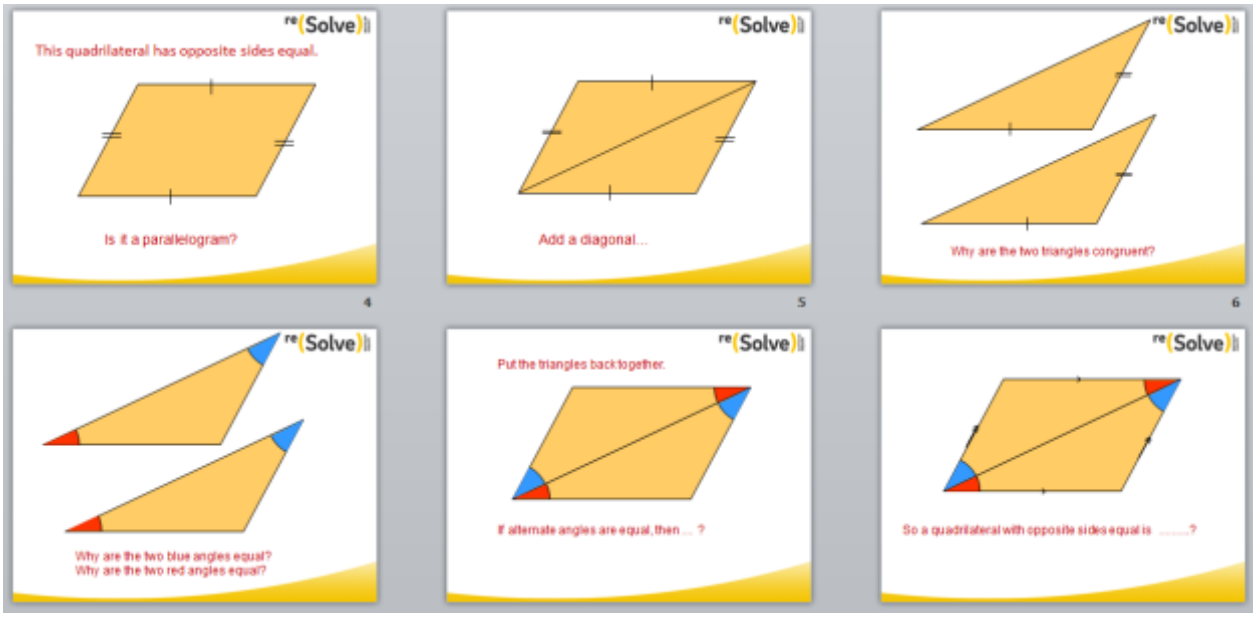
Expected student responses

- The opposite sides of the parallelogram stay parallel as the umbrella is opened and closed.
- The umbrella frame folds neatly.

Reasoning about Quadrilaterals

- The slide show *ST1_Yr8_3c_IsItParallelogram.ppt* provides visual support to help students reason through the steps in showing that a quadrilateral with opposite sides equal also has opposite sides parallel and hence is always a parallelogram.
- Students could watch the slide show independently or in pairs, filling in the reasons behind each step. A teacher-led discussion may be better for students with less experience of deductive reasoning.
- Visual reasoning through the slide show will be adequate for most students. More advanced students could write out their reasoning using this labelled diagram.
- Formal proofs are set out in Teacher Sheet 1 – Parallelograms and Deductive Reasoning for reference.





Conclusion

Conclude the lesson by revisiting the main ideas.

- If both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- The parallelograms in the ribs allow the umbrella to fold neatly as the opposite sides remain parallel.

Further Investigations

Constructing a 'drag-resistant' parallelogram and rectangle

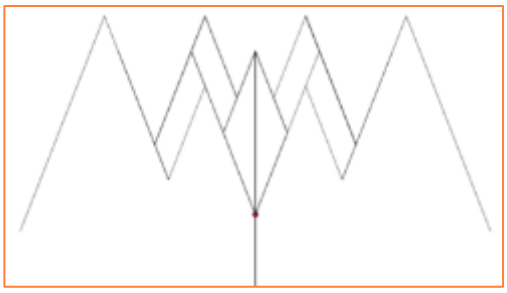
- Three construction tasks requiring students to use dynamic geometry software such as GeoGebra will encourage reasoning about which of the software tools they can use. A 'drag-resistant' shape remains the required shape when a vertex is dragged (e.g., a rectangle remains a rectangle).
    - Make a drag resistant parallelogram.
    - Make a drag-resistant rectangle.
    - Make a drag-resistant rectangle with one a vertex at the point (3,2) and the diagonally opposite vertex at the point (7,9).
- In each case students should have their figure tested for drag-resistance by a fellow student, and list the software tools that were used, for example, point, line, parallel line, circle.

Students will find different ways of completing their constructions. For the first two tasks, for example, some may use the perpendicular line tool whereas others may use a combination of parallel and perpendicular lines. Students frequently start by constructing a rectangle using only the visual appearance, for example, aligning the sides with the edges of the screen. The unexpected change of shape when this figure is dragged can then prompt a geometric construction based on rectangle properties.

Investigating linkages in other folding umbrellas

Some umbrellas fold to one third of their length. The ribs of these umbrellas have a double parallelogram linkage, as shown.

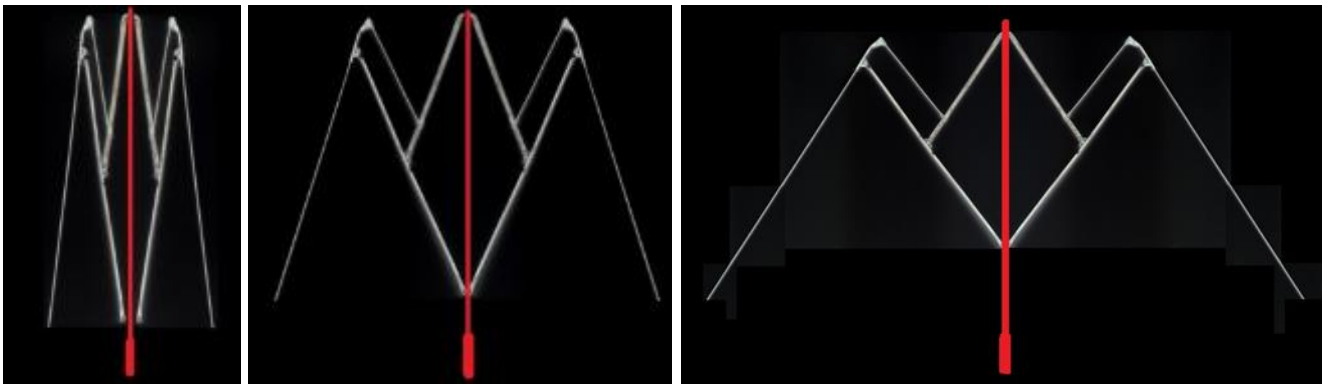
Students could look for different umbrellas, examine how they work, and see how the parallelogram linkages are used in other designs.



Folding Umbrella

Name: _____

The metal frames of most folding umbrellas have eight ribs that radiate out from the centre. Each rib is made up of hinged bars that allow the umbrella to fold.



- Examine one of the ribs of the folding umbrella as you open and close the umbrella.
- Identify a quadrilateral shape in each rib that appears to be a parallelogram (apart from a small bend at one vertex)
- What do you observe about the sides of this quadrilateral as the umbrella rib is opened and closed?

Making Physical Models

- Using 4 strips (2 of one length and 2 of a different length) and 4 paper fasteners, construct a quadrilateral with both pairs of opposite sides equal.
- As you change the shape of the quadrilateral, what do you notice about the opposite sides?
- Now use strips and paper fasteners to make a model of an umbrella rib.

Using a computer simulation

- Open the GeoGebra file *Umbrella*.
- Open and close the umbrella by moving the red point up and down.
- Observe how the parallelogram in each rib changes shape.
- How does this enable the umbrella to fold neatly?

Parallelogram Properties

- Now watch the slide show *Is it a Parallelogram*?
- Why is a quadrilateral with opposite sides of equal length always a parallelogram?

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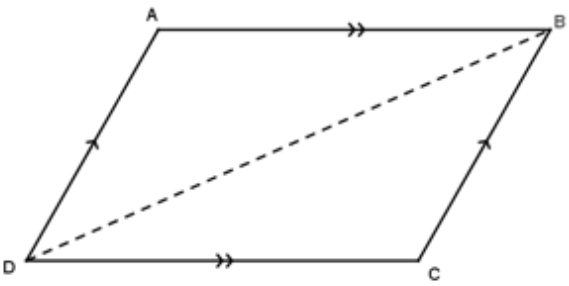
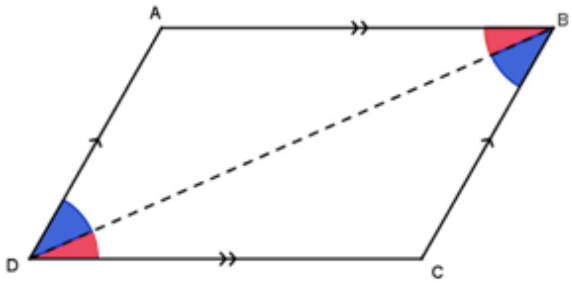
Teacher Sheet 1 - Parallelograms and Deductive Reasoning

Showing that Both Pairs of Opposite Sides of a Parallelogram are Equal in Length

$ABCD$ is a parallelogram. Hence AB is parallel to DC and AD is parallel to BC.





Prove: $AB = DC$ and $AD = BC$.

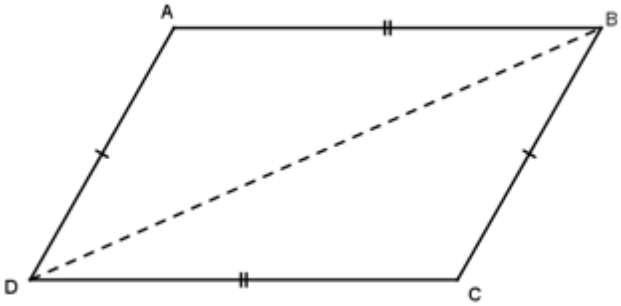
Proof: Draw the diagonal BD
In ΔABD and ΔCDB,
BD is common
$\angle ABD = \angle CDB$
$\angle ADB = \angle CBD$
So $\Delta ABD \equiv \Delta CDB$ (A.S.A.)
So $AB = DC$ and $AD = BC$

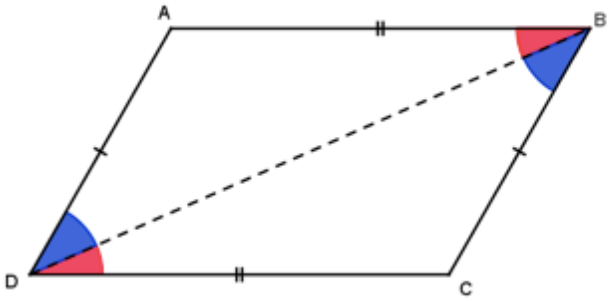
Showing that the Converse is True

A quadrilateral with both pairs of opposite sides parallel is a parallelogram because this is the definition of a parallelogram. We have shown above that the opposite sides of a parallelogram are equal.

However, can we assume that the converse is true? In other words, is a quadrilateral with both pairs of opposite sides equal necessarily a parallelogram? Yes!

$ABCD$ is a quadrilateral. $AB = DC$ and $AD = BC$. Show that $ABCD$ is a parallelogram.





Draw the diagonal BD
In ΔABD and ΔCDB,
BD is common
$AB = DC$ and $AD = BC$
So $\Delta ABD \equiv \Delta CDB$ (S.S.S.)
So $\angle ABD = \angle CDB$ (matching angles in congruent triangles)
So AB is parallel to DC (alternate angles are equal)
Similarly $\angle ADB = \angle CBD$ (matching angles in congruent triangles)
So AD is parallel to BC (alternate angles are equal)
Hence $ABCD$ has opposite sides parallel and so it is a parallelogram.

This proves that a quadrilateral with both pairs of opposite sides equal is a parallelogram.

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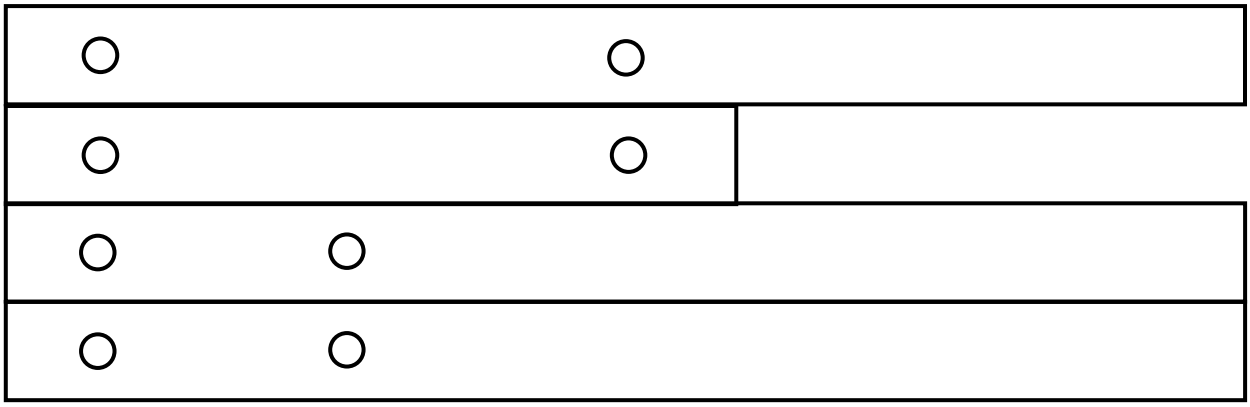
Folding umbrella strips template

The first set of strips is for making a quadrilateral with opposite sides equal. Cut out each strip and use 4 paper fasteners to make a quadrilateral with opposite sides equal.

The second set of strips is for making a model of the umbrella rib linkage. Cut out each strip and use 4 paper fasteners to make a model of an umbrella rib.