Year 10: Proof
Lesson 3: Pascal's Angle Machine

Australian Curriculum: Mathematics (Year 10)

ACMMG243: Formulate proofs involving congruent triangles and angle properties.

- Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).

ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

- Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
- Communicating a proof using a sequence of logically connected statements.

Lesson abstract

Students investigate the design and operation of an angle trisector invented by mathematician Blaise Pascal. The lesson has been deliberately named *Pascal's angle machine* rather than *Pascal's angle trisector*, so that students first explore the angle machine to find out what its purpose is, and then how it would be used. They use physical models and computer simulation to explore. Students then use the geometry of isosceles triangles and exterior angles to prove why the machine works.

Mathematical purpose (for students)

A sequence of logically connected geometric statements can explain why Pascal's angle machine works.

Mathematical purpose (for teachers)

The geometry of Pascal's angle machine is based on the interior and exterior angles of isosceles triangles. Students make conjectures, then identify relationships between angles in order to show why Pascal's angle machine is an angle trisector. For extension, students can investigate the long and rich history of the angle trisection problem. Alternatively they can generalise to more complex machines, an investigation which can reinforce the ideas of the lesson or lead to proofs to challenge the best students.

Lesson Length 50 minutes approximately

Vocabulary Encountered

- exterior angle
- trisect

Lesson Materials

- Per pair of students: 2 identical short strips and 2 long strips (e.g. 2 short red and 2 long red Geo Strips), 3 paper fasteners
- Slide show: *ST1_Yr10_3a_Pascal.pptx*
- YouTube video: https://youtu.be/T11ws1_tZWs
- GeoGebra file: *ST1_Yr10_3b_Pascal.ggb* OR GeoGebra Tube https://ggbm.at/KgmtBJ4y
- Student Sheet 1 – Pascal's Angle Machine (1 per student)

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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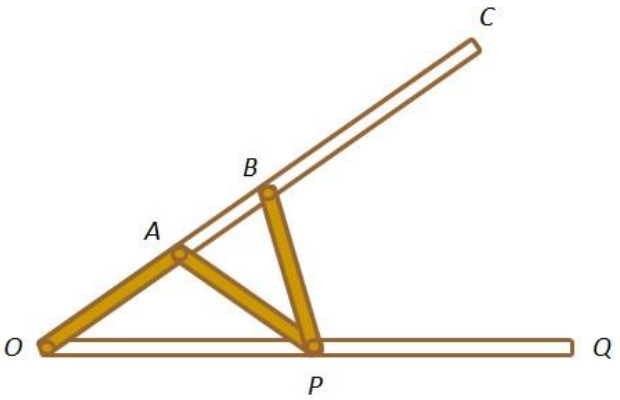
Introducing Pascal's Angle Machine

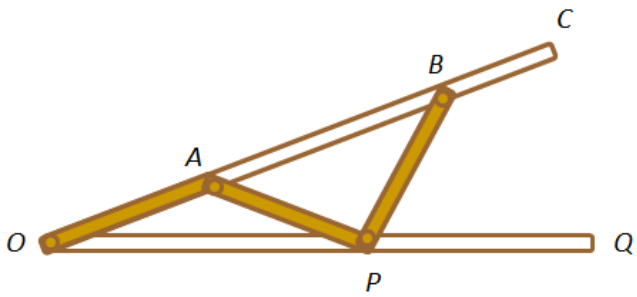
Teacher Notes

The task has been deliberately named *Pascal's angle machine* rather than the usual name of *Pascal's angle trisector*. Students explore its operation to find out what it does and then prove why it works.

The question of how to trisect an angle exactly is a very famous mathematical problem, first asked by the ancient Greeks. They knew it is easy to bisect an angle, using a ruler (straight edge) and compass, but they could not find a method that will trisect any angle, using these tools alone. The impossibility of trisecting an angle was finally proved by Pierre Wantzel in 1837. Mathematicians invented many ways to trisect angles with other tools.

In the 17th century the French mathematician Blaise Pascal invented an instrument for trisecting angles. In Pascal's angle trisector, $OA=AP=PB$, rods OC and OQ are hinged at O, and AP is hinged at A to OC. Point P can slide along the slot in the bar OQ, and point B slides along the slot in AC. To trisect an angle, the linkage is moved so that the angle to be trisected is aligned with $\angle BPQ$. The $\angle BOP$ is then one third of this.





Setting the challenge

- Show Students the Youtube video: https://youtu.be/T11ws1_tZWs or the first images from sideshow *Pascal's angle machine*.
- Explain that the challenge for this lesson is that students to find out what this machine is for, and why it works.
- Have students share initial observations about the linkage, such as
  - All the angles change when the arms are moved.
  - ΔOAP and ΔAPB are isosceles triangles because the sides OA, AP and PB are equal.
  - $\angle AOP = \angle APO$ and $\angle PAB = \angle PBA$
  - As $\angle AOP$ decreases, $\angle APB$ increases and OP becomes longer.
- Distribute Student Sheet 1 – Pascal's Angle Machine if desired.

Make a physical model

- Using Geo Strips, two long red strips, OC and OQ, are connected at O with a paper fastener. One of the short red strips, AP, is connected at A to OC so that an isosceles triangle is formed. The second short strip, PB, is only connected to AP at P (forming a second isosceles triangle) so that P is free to slide along OQ and B is free to slide along OC. Instructions for students are found on Student Sheet 1 – Pascal's Angle Machine.
- Students maniplate their models, and make, test and share conjectures about the purpose of the machine.

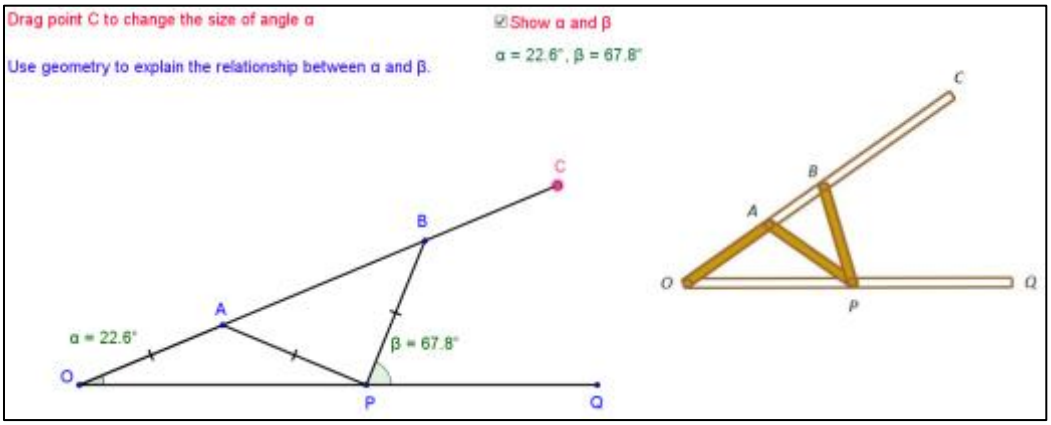
Teacher Notes

Ideally a model with slotted bars along which points B and P can slide could be made (e.g. using corflute), but a simple version of the linkage can be constructed using Geo Strips. This gives students a feel for the way the linkage moves, although mathematically more is to be gained from the computer simulation coming next, and the purpose of the linkage becomes easier to see.

Moving to the Mathematics

Using a computer simulation

The GeoGebra file *ST1_Yr10_3b_Pascal.ggb* provides a simulation which students can operate to see how the device works. The angle measures shown will assist students with conjecturing. As point *C* is moved, the angles change, but $\angle AOP$ is always one third of $\angle BPQ$. By observing the changing angle sizes, students should notice this relationship.



Constructing a Proof

There are several slightly different proofs. The most elegant is to recognise $\angle PAB$ as an exterior angle to ΔAOP (hence twice $\angle AOP$) and then use the fact that $\angle BPQ$ is an exterior angle to ΔBOP (hence three times $\angle AOP$). For details, see Teacher Sheet 1 – Pascal's Angle Machine Proof.

Enabling prompts

- Let $\angle AOP$ be α. What is the size of $\angle PAB$ in terms of α? Explain.
- What is the size of $\angle ABP$ in terms of α? Explain.
- For which triangle is $\angle BPQ$ an exterior angle?
- What is the size of $\angle BPQ$ in terms of α? Explain.
- Do you think the angle trisector could have worked if Pascal's design was not based on isosceles triangles?

Conclusion

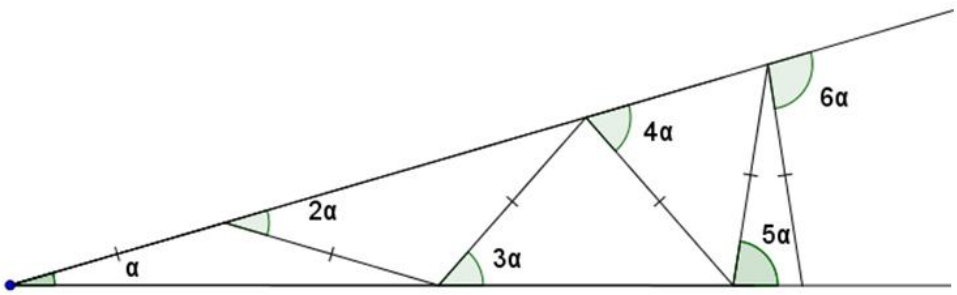
- Discuss students' proofs. Sharing first in small groups to refine, and then presenting to the class is one possibility.
- Pascal's angle trisector works because of the relationship between the interior and exterior angles of these carefully arranged isosceles triangles.

Extensions

Extending Pascal's machine

- Explore what happens if further links are added, following the same pattern?
- What could such a machine be used for, and how would it be used?
- Is there a limit to the number of links that can be added? If so, what is it?

Ans: The diagram below shows the machine with 5 links (so 6 sides of isosceles triangles).
This machine can be used to divide an angle by 2, 3, 4, 5, or 6.



There is a limit to the number of links that can be added. Another link can be added only if the angle $n\alpha < 90^\circ$ otherwise the next link will not be long enough to reach the other arm of the machine to follow the pattern. (Some students may be able to prove this by noting that when the angle is 90°, the arm becomes a tangent to a useful circle.)
In a GeoGebra construction we can continue adding links until we have about 10 or 11. Beyond that, the angle α needs to be very small to satisfy $n\alpha < 90^\circ$. With a physical model, the thickness of the links becomes the limiting factor quite soon.

History 1

- Investigate the history of the famous problem of trisecting an angle. There are many good references available.
- Why can't a diagram like Pascal's machine be constructed with a straight edge and compass, so that angles can be trisected? Why is it not possible to start with the angle, and work backwards to find the angle between the vertex of the arms?

History 2

- Construct a dynamic geometry file of Archimedes' angle trisection method. See Teacher Sheet 2 - Archimedes' Angle Trisection.

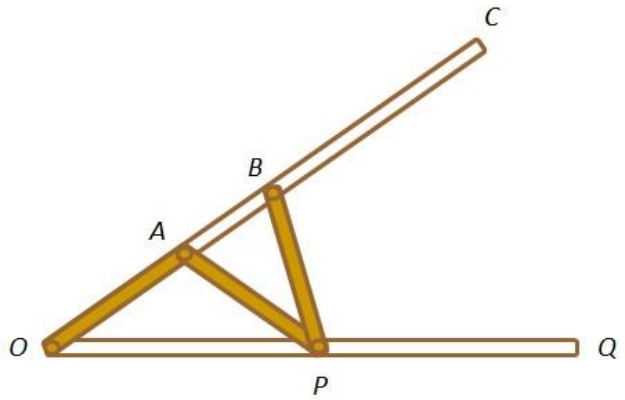
Pascal's Angle Machine

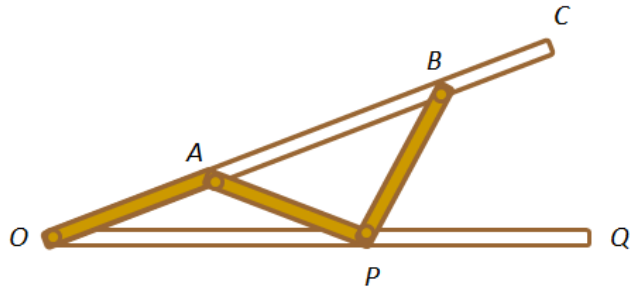
Name: _____

In the 17th century the French mathematician Blaise Pascal invented an angle machine. We want to work out what it was for.

In this angle machine:

- *OA*=*AP*=*PB*
- Rods *OC* and *OQ* are hinged at *O*
- *AP* is fixed to *OC* at *A*
- Point *P* can slide along a horizontal slot in the bar *OQ* at the base of the instrument, and point *B* can slide along a slot in *OC*.





Your task is to find out the purpose of Pascal's angle machine and then to prove why it works.

Making a physical model

Construct a model of Pascal's angle machine from strips such as Geo Strips. This model will not be perfect because the strips do not have slots in them.

- Connect two long red strips, *OC* and *OQ* with a paper fastener.
- Connect two short red strips, *AP* and *BP* with a paper fastener.
- Connect *AP* to *OC* at *A*. Make sure that *OA*=*AP*. You do not need to connect *P* or *B*. This is because *P* and *B* need to be able to slide.

- Make a list of what changes as you slide *P* along *OQ* (keeping *B* touching *OC*).
- What do you think this machine is for, and why does it work?

Using a computer simulation

- Open the GeoGebra file *Pascal*.
- Make a conjecture about the purpose of Pascal's angle machine.

Proving your conjecture

- It may help to use a letter such as α for $\angle AOP$ then try to express other angles in terms of α.

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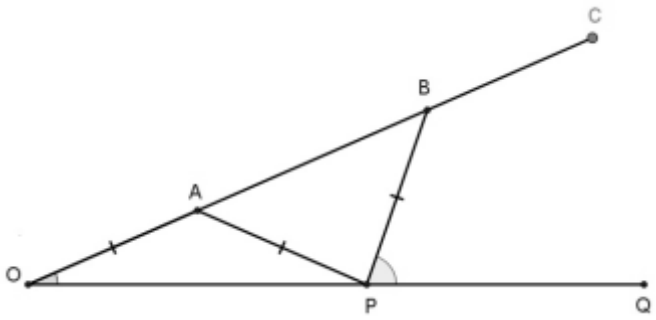
Teacher Sheet 1 – Pascal's Angle Machine Proof



Proof

Proving why the trisector works requires knowledge of isosceles triangles and exterior angles of triangles. In Pascal's angle trisector, $OA=AP=PB$ so that ΔOAP and ΔAPB are isosceles. Rods OC and OQ are hinged at O. AP and OC are hinged at A. As OC is rotated to change the size of $\angle BPQ$, B slides along OC and P slides along OQ. The proof that $\angle AOP$ is one third of $\angle BPQ$ is based on exterior angles of triangles.

In ΔAOP,

$\angle AOP = \angle APO = \alpha$ (base angles of isosceles triangle AOP)

$\angle BAP = \angle AOP + \angle APO$ (exterior angle)

$= 2\alpha$

For ΔBOP,

$\angle PBO = \angle BAP$ (base angles of isosceles triangle PAB)

$\angle BPQ = \angle PBO + \angle BOP$ (exterior angle)

$= 2\alpha + \alpha$

$= 3\alpha$

$\angle BPQ = 3\angle BOP$

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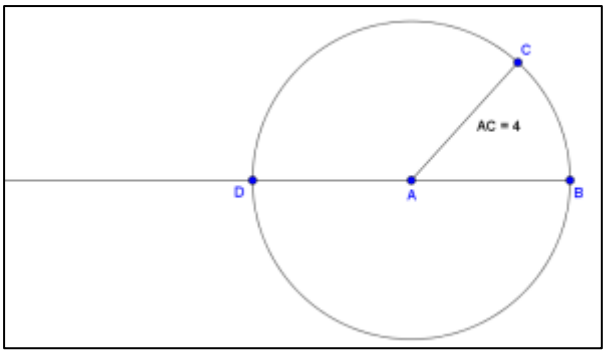
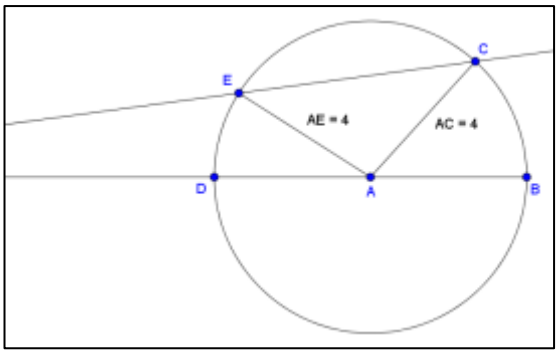
Teacher Sheet 2 – Archimedes' Angle Trisection

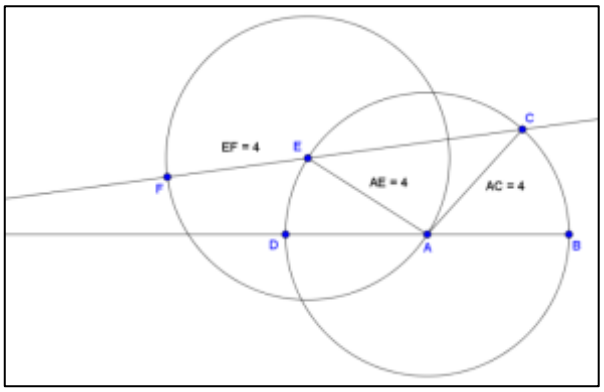
Trisecting an angle was a famous problem of classical Greek geometry. Euclid showed in about 300 BCE that it was possible to bisect an angle using straight edge and compass. In 1837 the French mathematician Wantzel showed that trisection of a general angle is not possible using straight edge and compass. However, early Greek mathematicians found other ways of trisecting angles. One method, attributed to Archimedes, survived in a medieval Arabic translation. The diagram below shows some resemblance to Pascal's angle trisector linkage.

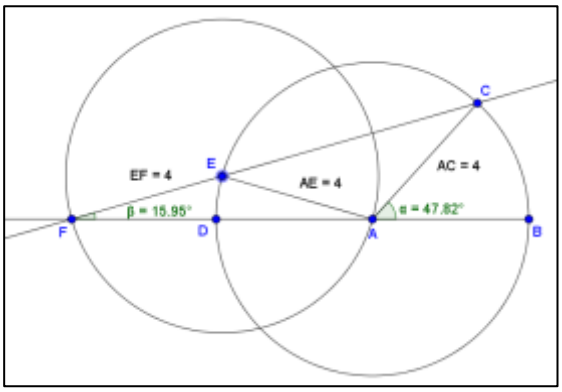
Using GeoGebra to replicate the method, we start with a circle, centre *A*, with diameter *BAD*. *C* is a point on the circle, with ∠*CAB* being the angle to be trisected. An arbitrary point *E* is placed on the circle, then the line *CE* drawn. In Archimedes' method, *CE* represents a straight edge marked with the radius of the circle.



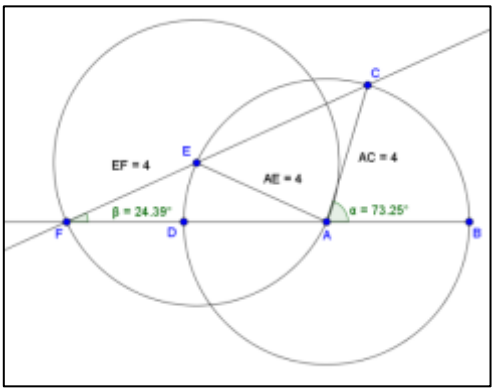


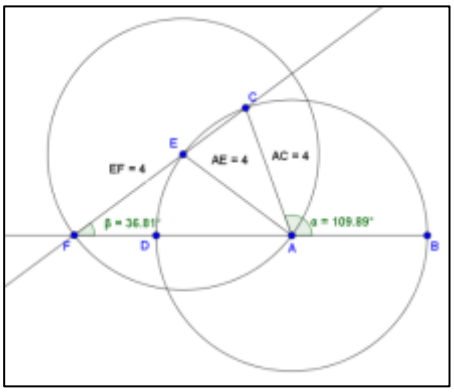
A point *F* is placed on the line *CE* extended so that *EF* is equal to the radius *AB* = *AC* = *AE*. Point *E* is then dragged around the circle so that *F* meets the line *BD* extended. This cannot be done exactly, but ∠*EFD* is very close to one third of ∠*CAB*. In Archimedes' method, the straight edge would be slid and turned (while still touching *C*) until the distance between *E* and *BD* produced matched the marked radius.





In our simulation we can now move *C* around the circle and watch the angles change.





The reasoning behind this is of course the same as for Pascal's angle trisector: exterior angles of the isosceles triangles. However, whereas the above GeoGebra construction and the method used by the early Greeks requires an element of approximation, Pascal's angle trisector is exact because the points are constrained by the physical linkage construction.

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