Year 10: Proof
Lesson 4: Consul

Australian Curriculum: Mathematics (Year 10)

ACMMG243: Formulate proofs involving congruent triangles and angle properties.

- Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).

ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

- Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
- Communicating a proof using a sequence of logically connected statements.

Lesson abstract

Students construct a physical model of Consul, a toy calculator, and use a computer simulation of Consul to explore how the geometric design enables the product of numbers between 2 and 12 to be displayed. Students may then develop a proof based on a sequence of deductive reasoning.

Mathematical purpose (for students)

Geometry is used in the design of many mechanical objects. We can use a sequence of logically connected geometric statements to explain why the object works the way it does. In order to develop a sequence of reasoning to explain why the object works, we sometimes need to add construction lines to a geometric diagram.

Mathematical purpose (for teachers)

The lesson provides a novel and challenging way to engage students in geometric reasoning, leading to construction of a geometric proof. The geometry of Consul is based on angle and side properties of a rhombus and congruent isosceles triangles. Students must investigate relationships between angles and between sides in order to identify the geometric property that is critical to the functioning of Consul. Exploration with the dynamic geometry file will be essential. In addition, students will need a very clear sense of what they are trying to prove and why.

Lesson Length 100 minutes approximately

Vocabulary Encountered

- similar triangles
- construction line

Lesson Materials

- Consul toy calculator if possible (available from many websites)
- Slide show: *ST1_Yr10_4a_ConsulImages.pptx*
- Geo Strips or similar (per group, 6 identical strips and 2 strips that are √2 times as long; 6 paper fasteners)
- Multiplication Number Grid (1 per pair on A4 paper or thin card)
- Addition Number Grid (optional, on A4 paper or thin card)
- Student Sheet 1 – Consul (1 per student)
- GeoGebra file: *ST1_Yr10_4b_Consul.ggb* OR GeoGebra Tube https://ggbm.at/UHa7hVTj
- Access to Consul website or downloaded files (optional, PC only)

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

aamt

Introducing Consul

Teacher Notes

'Consul', the Educated Monkey is an American-designed toy, invented in 1916 to teach children multiplication tables, factors and simple addition and subtraction. Modern replicas are readily available. The tin toy is based on a linkage which is designed so that when the feet are set to point to two numbers, the hands point to the product. The multiplication card behind the monkey can be replaced by an addition card.

A replica simulation of *Consul* is available at the website: http://www.rechenwerkzeug.de/

Alternatively, click on the link Consul für Windows to download the zipped file (PC only), which must then be unzipped to give the replica simulation shown below.

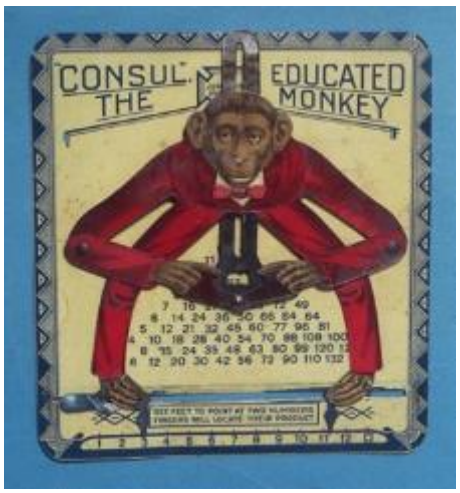
How does Consul work?

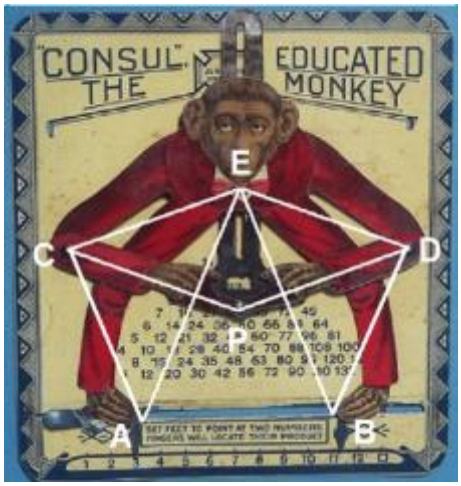
Each upper arm and leg is constructed from a single piece of tin plate so that the triangles ΔACE and ΔBDE are rigid and do not change shape. They are made as congruent isosceles triangles with $\angle ACE$ and $\angle BDE$ as right angles.

These two pieces of tin pivot about a point (E) beneath the bow tie.

Segments CE, DE, CP and DP are all equal, so that $CEDP$ is a rhombus. The rhombus changes shape via hinges at points C and D (the monkey's elbows) and at points E and P. The linkage remains symmetrical about the vertical line through EP. The slotted tail ensures that EP stays vertical, which is essential if P is to be located above the product of the two numbers.

The feet, at A and B, can move only in a straight line through A and B, but can be positioned on any pair of numbers. Point P moves along the multiples of the numbers at which the feet are set (the grid of numbers is made this way) so that it is always at the product of the two numbers. To find the square of a number, place one foot on the number and the other foot on the square next to the 12.





Getting started

- Show students the start of the slide show *ST1_Yr10_4a_ConsulImages.pptx*, or a simulation (listed above), and ask for observations on what Consul is doing, and how it works.

Expected Student Response

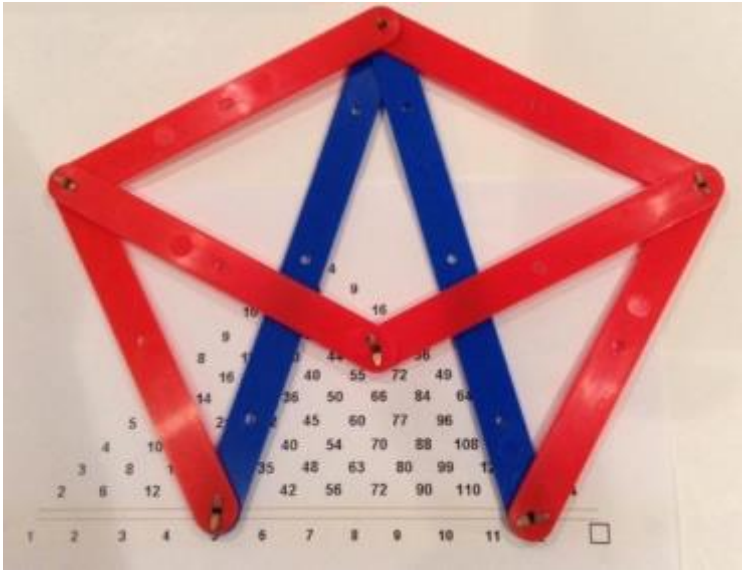
- If the monkey's feet are set on two numbers, the monkey's fingers always point to the product of the two numbers.
- Possibly some students will make conjectures about the angles and lengths, but this is not expected until the construction is shown.

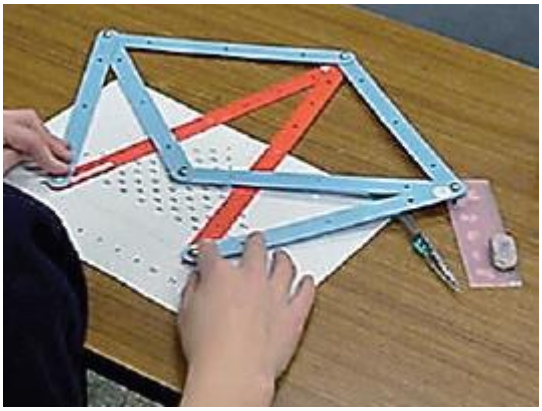
Models of Consul

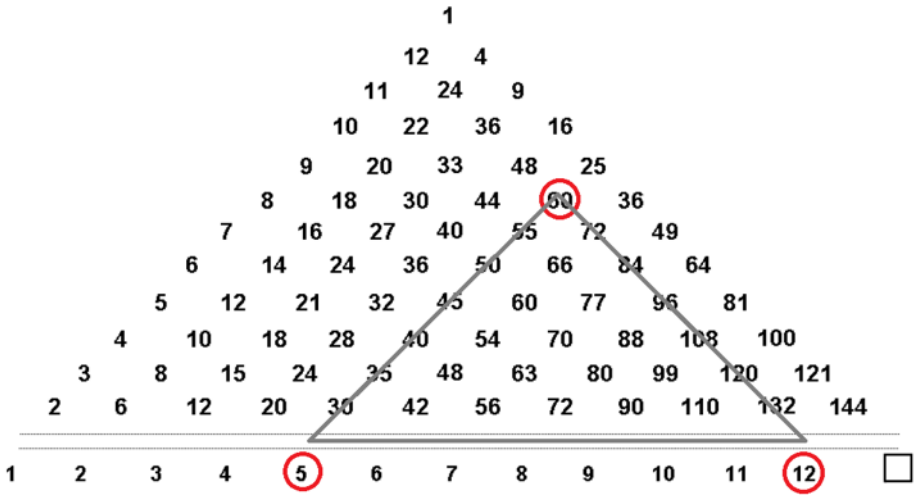
Making a physical model

Students can construct a model of the Consul from Geo Strips and paper fasteners using two long blue Geo Strips and six of the medium red strips. The length of a long blue strip is √2 times the length of the medium red strip so a right-angled isosceles triangle can be constructed from them. The Multiplication Number Grid should be placed under the Geo Strip linkage, with the 'feet' (the paper fasteners) between the dotted lines on the grid.

Students begin to observe the geometry (equal lengths, right angles, rhombus).

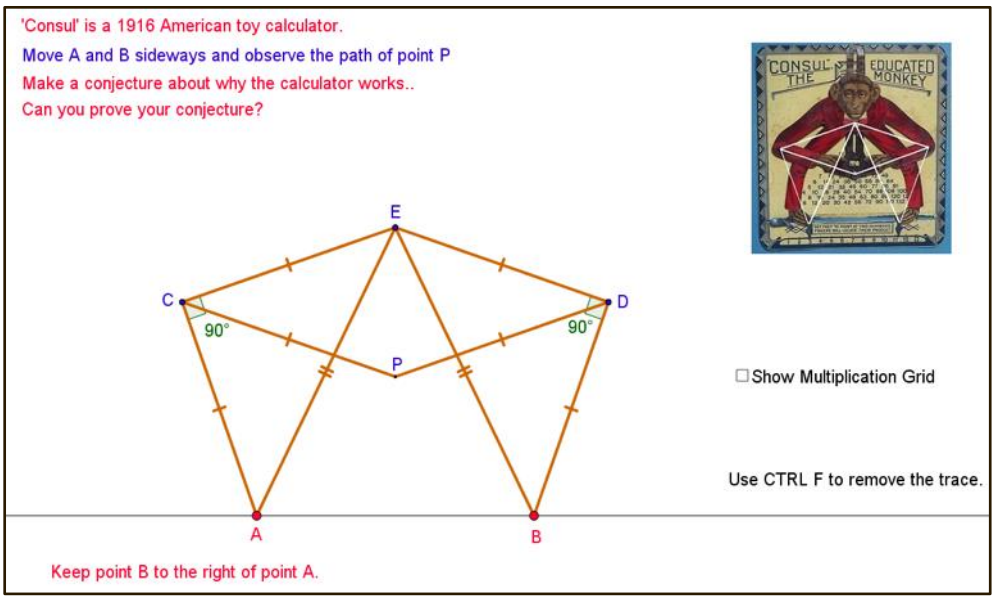




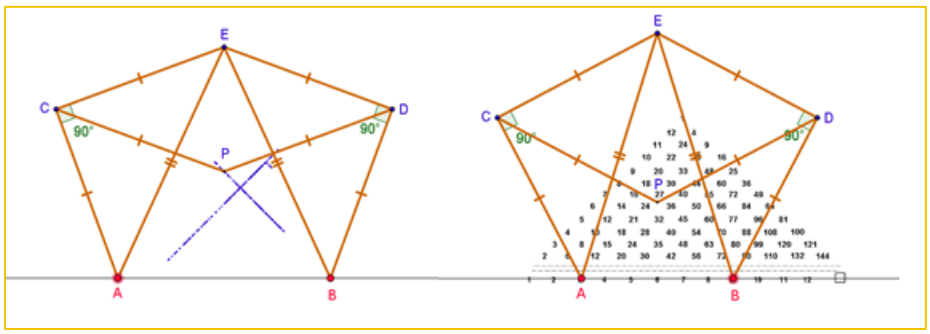


Using a computer simulation

- The GeoGebra file *ST1_Yr10_4b_Consul.ggb* is a simulation of the calculator. Students can change the positions of the feet, *A* and *B* (preferably one at a time) and observe the traced path of point *P*.
- Note that the trace may be under the grid.

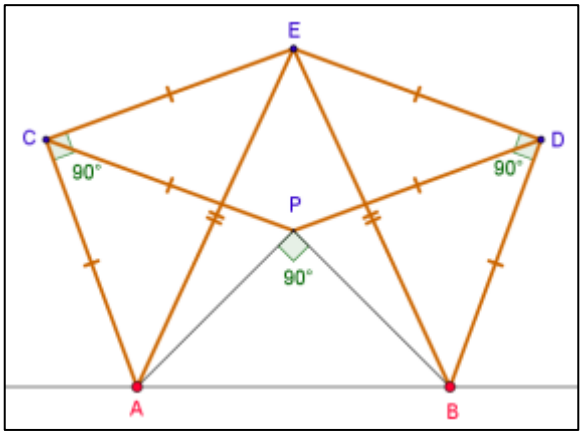


- Students are likely to observe that the trace of point P forms two lines at right angles as the feet are moved separately.
- When the number grid is shown, students are likely to observe that point P moves along the diagonal number rows.



- Working in pairs, students should be encouraged to develop their reasoning in the dynamic environment of the computer simulation. They may notice properties that stay constant and properties that vary as they move point *A* or point *B*, for example, they may conjecture that ΔACP and ΔAEB are similar (and ΔBDP).

- If students are working with the *Consul* GeoGebra file rather than with the GeoGebra tube version, then they may choose to insert segments *AP* and *BP* and perhaps measure ∠*APB*. If they do this they will find that ∠*APB* = 90°.
- After some exploration students are likely to observe that the numbers in the number grid are arranged in a right-angled triangle and that point *P* moves down the lines of numbers that are multiples of the numbers on which the feet are placed. Point *P* will always be at the common multiple of the two numbers on which the feet are placed.
- With the conjecture that ∠*APB* = 90°, the task then is to use the geometry of the linkage to prove why this angle stays a right angle.



Constructing proofs

The same unstructured approach has been taken to this lesson as for the *Sylvester's Pantograph* and *Peaucellier's Linkage* investigations. By now, students should be developing an understanding of conjecturing and proving and how a proof is constructed from a series of steps of reasoning which are justified on the basis of given information or previously accepted understanding.

- Exploration with the GeoGebra file seems essential to support this investigation. Observation of movements, and experimentation and measurement provide ideas and evidence on the truth of conjectures.
- Drawing a careful diagram and labelling with the same letters for angles that are known to be equal helps students to develop a logical sequence of reasoning.
- Some students can engage in sustained discussion and reasoning with very little input from the teacher. Teachers can encourage this in others through pair work or group work.

Some students will be assisted with a series of questions:

- What geometric shape is quadrilateral $CEDP$? Explain.
- From the given information what do you know about ΔAEB?
- From the given information what do you know about ΔACP and ΔBDP?
- What can you say about the four angles $\angle CEA$, $\angle EAC$, $\angle DEB$ *and* $\angle EBD$? Explain.
- Show that $\angle AEB = \angle ACP$ and hence that ΔAEB is similar to ΔACP
- What can you now say about $\angle APB$?

Conclusion

- The Consul toy calculator works because the right-angled isosceles triangle of the linkage corresponds with the right-angled isosceles triangle arrangement of the numbers in the multiplication grid. Starting with known facts and knowledge of geometric properties, a geometric argument can be constructed as a sequence of logically connected statements. It can be shown that the triangle formed by the monkey's feet and the point where the product of the two numbers is displayed is a right-angled isosceles triangle..

Extensions

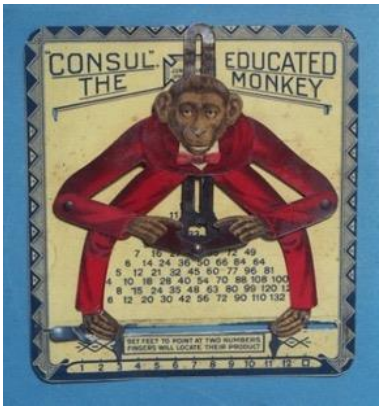
- Design a number grid so that Consul can be used for addition. What about subtraction? What about other operations (e.g. twice the first number add the second)?
- How do the different models of Consul deal with square numbers? Why do the square numbers need special treatment? (ANS: the two feet have to be in the same place). Some versions have the square numbers in the lower row. The versions in the GeoGebra file supplied and the model above have the square numbers down the right side, and they are reached by placing one foot on the number and the other foot on the Square next to the 12.

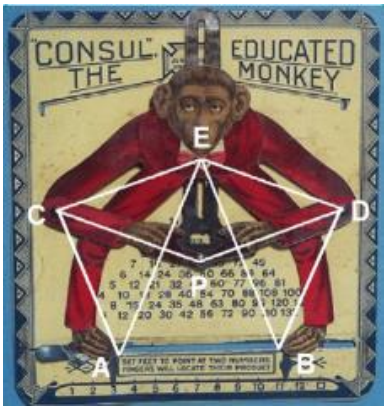
Consul

Name: _____

'Consul', the Educated Monkey is an American-designed toy, invented in 1916 to teach children multiplication tables, factors and simple addition and subtraction. The tin toy is based on a linkage which is designed so that when the feet are set to point to two numbers, the hands point to the product. The multiplication card behind the monkey can be replaced by an addition card.

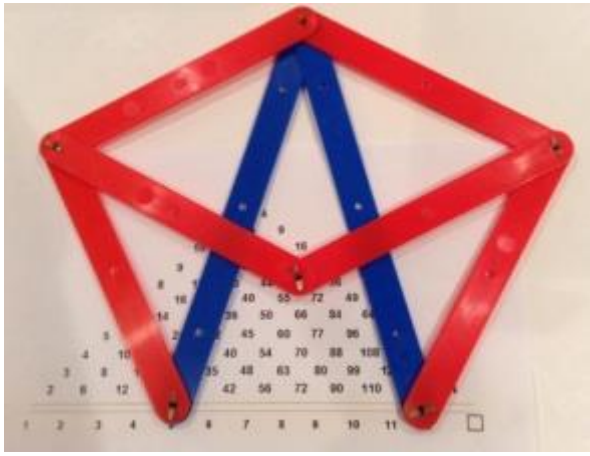
- Each upper arm and leg is constructed from a single piece of tin plate so that ΔACE and ΔBDE are right angled isosceles triangles that do not change shape. These two pieces of tin pivot about a point (E) beneath the bow tie.
- Segments CE, DE, CP and DP are all equal, so that $CEDP$ is a rhombus. The rhombus changes shape via hinges at points C and D (the monkey's elbows) and at points E and P.
- The feet, at A and B, can move only along the straight line through A and B, but can be positioned on any pair of numbers.
- The slotted tail ensures that the linkage is always symmetrical about the vertical line through EP.





Making a physical model

- Construct a model of Consul from 4 identical strips, and 2 longer strips which are 1.4 times the length of the 4 shorter strips. You will need 6 paper fasteners.
- Place the number grid under the linkage with the feet, A and B, on the dotted lines. Move the feet so that they are directly above two numbers on the number line. The product of the numbers should be just above P.
- Move the feet to a different pair of numbers and observe how the linkage moves on the number grid.
- Challenge: 1.4 is approximately $\sqrt{2}$. Why is this necessary?



Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

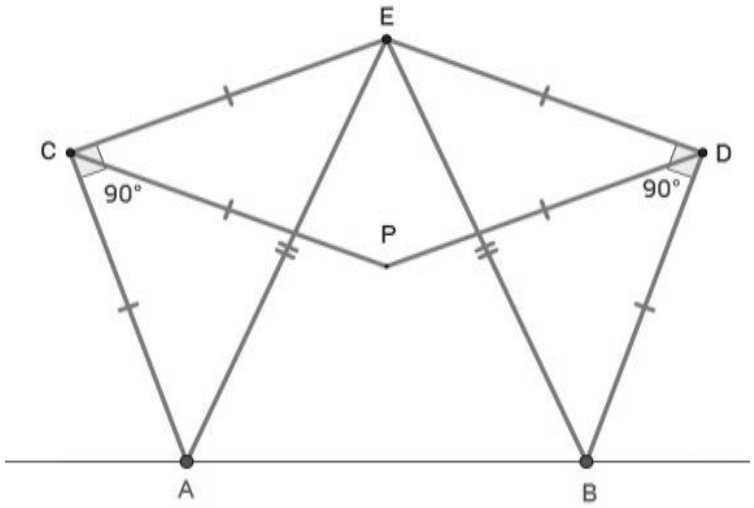
aamt

Using a computer simulation

- Now open the GeoGebra file *Consul* and drag point *A* then point *B*. Use the simulation to try to work out why the linkage works. Moving one point at a time and leaving 'traces' is a good strategy.
- Make a conjecture relating to the geometry of the linkage that would explain why it works.

Prove your conjecture and explain why Consul works

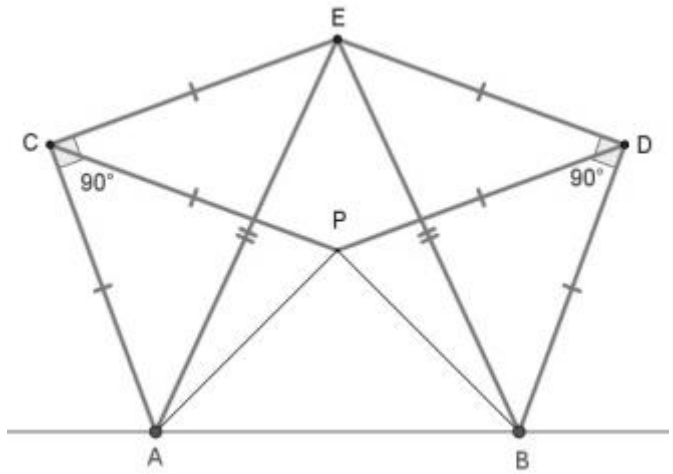
- Using the following diagram, set out a proof of your conjecture, justifying each statement you make. You may need to construct extra lines on the diagram.



- Explain in words how the geometry of the toy allows it to work

.

Teacher Sheet 1 - Consul Proof



Outline of proof that $\angle APB = 90^\circ$

This proof uses two sets of similar triangles to show that $\angle APB = \angle ACE = 90^\circ$

Prove ΔACP and ΔAEB similar

$\angle ACP = 90^\circ - \angle PCE$

But $\angle PCE = 180^\circ - \angle CED$ (rhombus $CEDP$)

$= 180^\circ - (45^\circ + 45^\circ + \angle AEB)$

$= 90^\circ - \angle AEB$

Hence $\angle ACP = 90^\circ - (90^\circ - \angle AEB)$

So $\angle ACP = \angle AEB$

ΔACP and ΔAEB are similar (ΔACP and ΔAEB are isosceles with $\angle ACP = \angle AEB$)

Prove ΔAPB and ΔACE similar

From above, ΔACP and ΔAEB are similar (S.A.S.)

So $\dfrac{AB}{AP} = \dfrac{EA}{AC}$

$\dfrac{AB}{EA} = \dfrac{AP}{AC}$

And $AP = PB$ (consul is symmetrical about EP, ΔACP and ΔBDP are congruent)

So ΔAPB and ΔACE are similar (all sides in same ratio)

Hence $\angle APB = 90^\circ$

The Consul toy calculator works because the right-angled isosceles triangle in the linkage corresponds with the right-angled isosceles triangle arrangement of the numbers in the multiplication grid.

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

aamt

1

12 4

11 24 9

10 22 36 16

9 20 33 48 25

8 18 30 44 60 36

7 16 27 40 55 72 49

6 14 24 36 50 66 84 64

5 12 21 32 45 60 77 96 81

4 10 18 28 40 54 70 88 108 100

3 8 15 24 35 48 63 80 99 120 121

2 6 12 20 30 42 56 72 90 110 132 144

1 2 3 4 5 6 7 8 9 10 11 12 $\square$

14

13 15

12 14 16

11 13 15 17

10 12 14 16 18

9 11 13 15 17 19

8 10 12 14 16 18 20

7 9 11 13 15 17 19 21

6 8 10 12 14 16 18 20 22

5 7 9 11 13 15 17 19 21 23

4 6 8 10 12 14 16 18 20 22 24

3 5 7 9 11 13 15 17 19 21 23 25

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**