re(Solve) MATHS BY INQUIRY

What you need to know: PROPORTIONAL REASONING

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Proportional reasoning is a way of thinking mathematically. Students who reason proportionally appreciate the multiplicative relationship between quantities or measures that is evident in situations of comparison (Behr, Harel, Post & Lesh, 1992). Proportional reasoning forms a substantial area of mathematics studied at school, yet many students do not exhibit the capacity to reason proportionally (Victoria Department of Education, 2017). Lamon (2012) extends this concern to adults, claiming that as many as 90% of adults do not reason proportionally, despite many real-world situations requiring such thinking.

Proportional reasoning is founded on key early mathematical concepts and serves as the foundation of mathematics in the middle years and beyond (Lamon, 2012). In the early years, place value, multiplication, division and fractions introduce ideas of proportionality (Van de Walle, 2015). In the middle years and beyond, concepts such as probability, scale, percent, trigonometry, equivalence, measurement, algebra and the geometry of plain shapes require students to be able to reason proportionally (Dole, Clarke, Wright & Hilton, 2012).

What is proportional reasoning?

> *Proportional reasoning refers to detecting, expressing, analysing, explaining, and providing evidence in support of assertions about proportional relationships. The word reasoning further suggests that we use common sense, good judgment, and a thoughtful approach to problem-solving, rather than plucking numbers from word problems and blindly applying rules and operations. We typically do not associate reasoning with rule-driven or mechanized procedures, but rather, with mental, free-flowing processes that require conscious analysis of the relationships among quantities.* (Lamon 2012, p. 4)

Many concepts in the *Australian Curriculum: Mathematics* draw on proportional reasoning skills. The capacity to reason proportionally stems from a conceptual understanding of ratio and proportion.

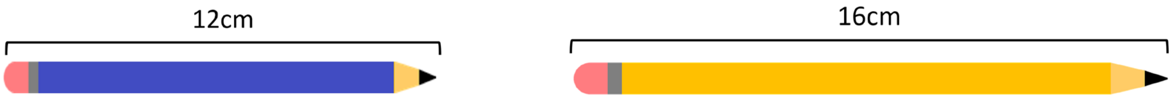
Ratio, rate and proportion

A ratio expresses a multiplicative relationship between two quantities or measures. Students' initial experiences with measures require reasoning with a single quantity (Lobato & Ellis, 2010). Ratios require students to attend to and coordinate two quantities simultaneously (NCTM, 2013).

Ratios can be expressed in one of two ways: a multiplicative comparison of two units or the joining of two quantities as a composed unit.

Multiplicative comparison

A ratio can be represented as a multiplicative comparison. Lobato & Ellis (2010, p. 18) state that "forming a multiplicative comparison involves asking, 'How many times greater is one thing than another?' or 'What part or fraction is one thing of another?'". In contrast, additive comparisons look at how much greater or smaller one thing is compared to another.



Additive comparison: The yellow pencil is 4cm longer than the blue pencil.

Multiplicative comparison: The blue pencil is ¾ the length of the yellow pencil.

Composed unit

A composed unit is the coordination of two quantities to form one new unit (Lamon, 1994). Van de Walle and colleagues illustrate composed units using the cost of buying kiwifruit: if kiwifruit are priced at 4 for $1, then 2 for 50c, 8 for $2 or 12 for $3 would also be true. New, equivalent units or ratios are formed by partitioning and iterating the original composed unit. Initially, students use intuitive partitioning and iterating strategies such as doubling and halving. More sophisticated skills are built over time through a variety of different problems (Ellis, 2013).

Proportion refers to equivalence between two ratios. Lobato & Ellis (2010, p.33) define proportion in the following way: "A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change".

> *Recognising and maintaining a proportional relationship within and between ratios is dependent on a sense of number. ...more than just learning and applying techniques and procedures... Students need to recognise the multiplicative relationship between the quantities involved and the capacity to recognise and maintain the proportional relationship as they work with the units involved.* (Siemon et al., 2011)

Rate

The proportional relationship between two units is described as the *rate*. Ratios have associated rates. Take, for example, mixing paint: the ratio 3 parts white paint for every 2 parts blue paint has the associated *rate* $\frac{3}{2}$ parts white paint for every 1 part blue paint.

Substantial Mathematical Ideas

reSolve uses the term *Substantial Mathematical Ideas* to identify key understandings that are critical to the development of a mathematical concept. We have identified five features that make a mathematical idea substantial:

- CONCEPTUAL: Focuses on understandings over procedures and skills
- STRUCTURAL: Focuses on the structure of mathematics
- CONNECTED: Connects mathematical concepts across domains
- TRANSFORMATIVE: Challenges and repositions existing conceptions
- GENERATIVE: One idea generates another

Substantial Mathematical Ideas for proportional reasoning

Lobato & Ellis (2010, p.11) identify one overarching mathematical idea for proportional reasoning:

When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor.

Van de Walle, Karp and Bay-Williams (2015) specify four additional substantial mathematical ideas in proportional reasoning:

A ratio is a multiplicative comparison of two quantities

A ratio expresses a multiplicative relationship between two quantities within a given situation. A ratio can express a part-to-part relationship, a part-to-whole relationship, a quotient, or a rate.

Ratios and proportions involve multiplicative comparisons

Equality between ratios is maintained through the use of multiplication and division, not addition and subtraction. Students need to recognise that addition and subtraction do not maintain equivalence.

Rate is a way to represent a ratio

A rate describes the proportional relationship between two different units, for example centimetres in a metre or litres per kilometre. Students need to appreciate that a rate is an infinite set of ratios (Lobato & Ellis, 2010).

Proportional reasoning involves comparing and determining equivalence of ratios

Proportional reasoning is a way of thinking mathematically. Students who can reason proportionally about quantities have the capacity to solve a wide variety of proportional problems through flexible use of strategies—they are not reliant on memorised rules and procedures.

Learning progression

Year 5 to Year 6	
What students will come to know and understand	Ratios represent an invariant multiplicative relationship between two units. Ratios can be a part-part relationship, e.g. 9 girls to 6 boys. Ratios can be a part-whole relationship, e.g. 9 girls in a class of 15. Ratios can be thought of as multiplicative comparison, e.g. how many times greater one thing is than another. Ratios can be thought of as a composed unit, e.g. 4 lemons for $1. In a ratio, the numerical values of both quantities need to change by the same factor for the proportion to remain constant. This is known as covariation. There are multiple ways to calculate with ratios and the efficiency of strategies is dependent on the context and/or numerical values of units. Proportional measurements have the same scale factor.
Indicators of understanding	Students recognise everyday situations that involve ratio, rate and proportion. Students identify proportional and non-proportional situations. Students use multiplicative strategies in proportional situations. Students use multiplication and division to calculate equivalent ratios. Students solve simple problems involving ratios and rates. Students use scale to accurately calculate measurements.

Year 7 to Year 8	
What students will come to know and understand	Ratios represent an invariant multiplicative relationship between two units. Ratios can be a part-part relationship, e.g. 9 girls to 6 boys. Ratios can be a part-whole relationship, e.g. 9 girls in a class of 15. Ratios can be thought of as multiplicative comparison, e.g. how many times greater is one thing than another. Ratios can be thought of as a composed unit, e.g. 4 lemons for $1. A ratio between two different units is a rate. A rate is a set of infinitely many equivalent ratios. In a ratio, the numerical values of both quantities need to change by the same factor for the proportion to remain constant. This is known as covariation. There are multiple ways to calculate with ratios and the efficiency of strategies is dependent on the context and/or numerical values of units. Ratios and fractions are related. A ratio that represents a part-whole relationship can be expressed as a percentage. The percentage represents the proportion compared to one hundred. Proportional measurements have the same scale factor. Congruence is a special case of similarity. Proportional situations are linear situations. Proportional relationships are linear and always pass through the origin.
Indicators of understanding	Students recognise everyday situations that involve ratio, rate and proportion. Students identify proportional and non-proportional situations. Students use multiplicative strategies in proportional situations. Students use multiplication and division to calculate equivalent ratios. Students use a variety of strategies for ratio and rate problems, including scaling up and down, ratio tables, and unit ratios. Students apply efficient strategies to solve problems involving ratios and rates. Students use scale to accurately calculate measurements. Students recognise whether a relationship will be linear or non-linear. Students plot points and graph linear relationships. Students recognise that a straight line that passes through the origin represents a ratio. Students identify and explain conditions for congruency of triangles and quadrilaterals. Students identify congruent triangles and quadrilaterals.

Year 9 to Year 10	
What students will come to know and understand	There are multiple ways to calculate with ratios and the efficiency of strategies is dependent on the context and/or numerical values of units. Two units are in direct proportion when an increase in one unit causes an increase in the other unit. Two units are in indirect (or inverse) proportion when an increase in one unit causes a decrease in the other unit. Similar figures are a visual representation of proportion. Congruence is a special case of similarity. Ratios are a special linear situation which always pass through the origin. The rate of change is represented by the slope of the line.
Indicators of understanding	Students recognise everyday situations that involve ratio, rate and proportion. Students identify proportional and non-proportional situations. Students use multiplicative strategies in proportional situations. Students use multiplication and division to calculate equivalent ratios. Students use a variety of strategies for ratio and rate problems, including scaling up and down, ratio tables, and unit ratios. Students apply efficient strategies to solve problems involving ratios and rates. Students use scale to accurately calculate measurements. Students recognise whether a relationship will be linear or non-linear. Students plot points and graph linear relationships. Students recognise that a straight line that passes through the origin represents a ratio. Students identify and explain conditions for congruency of triangles and quadrilaterals. Students identify congruent triangles and quadrilaterals.

References

Behr, M., Harel, G., Post, T. & Lesh, R. 1992. Rational number, ratio and proportion. In D. Grouws (Ed.), Handbook of Research in Mathematics Teaching and Learning (pp.296-333). New York: MacMillan.

Dole, S. 2008. Ratio tables promote proportional reasoning in the primary classroom. Australian Primary Mathematics Teachers, pp 16- 22, 13(2).

Dole, S., Clarke, D., Wright, T. & Hilton, G. 2012. Students' proportional reasoning in mathematics and science. In T. Y. Tso (Ed.) Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education (pp. 195-202). Taipei, Taiwan: PME.

Ellis, A. 2013. Teaching Ratio and Proportion in the Middle Grades. Retrieved June 6 2019, from https://www.nctm.org/uploadedFiles/Research_and_Advocacy/research_brief_and_clips/research%20brief%2021%20-%20Teaching%20Ratio%20and%20Proportion%20in%20the%20Middle%20Grades_Final(1).pdf

Jacob, B & Fosnot, C. (2007). Best buys, Ratios, and Rates: Addition and subtraction of fractions. Portsmouth, NH: Heinemann.

Lamon, S. (2012). Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies. New York, NY: Taylor & Francis Group.

Lobato, J., & Ellis, A.B. (2010). Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics, Grades 6–8. National Council of Teachers of Mathematics.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2011). Teaching Mathematics: Foundations to Middle Years. South Melbourne: Oxford University Press.

Victoria Department of Education. 2017. Common Misunderstandings - Level 5 Proportional Reasoning. Retrieved 6 June 2019 from https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/lvl5prop.aspx

Van de Walle, J., Karp, K. & Bay-Williams, J. (2015). Elementary and Middle School Mathematics: Teaching Developmentally, 9th Edition. Essex: Pearson Education.